A Theory of Liquidity and Risk Management Based on the Inalienability of Risky Human Capital*

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Abstract

We analyze a dynamic optimal financial contracting problem in continuous time with risky cash flows between a risk-averse entrepreneur and risk-neutral investors. Two fundamental constraints on the contracting parties are that: 1) the entrepreneur cannot alienate his human capital, and 2) investors have limited liability protection. Given that human capital is risky, the entrepreneur's inability to commit his human capital to the firm generates significant distortions for risk-sharing, corporate investment, and consumption. We show that the optimal contracting problem boils down to a corporate liquidity and risk-management problem for the firm. We also show that equilibrium default on a credit line is optimal when firms face persistent productivity shocks. Finally, we quantitatively value the net benefits of risk and liquidity management and find that they are high. Our analysis thus provides new foundations for liquidity and risk management policies that firms routinely pursue in practice.

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1 Introduction

Neither an entrepreneur in need of funding, nor anyone else for that matter, can legally agree to enslave himself to a firm in exchange for financing by outside investors. This simple observation has led Hart and Moore (1994) to formulate a theory of corporate external financial constraints stemming from the inalienability of human capital. In a highly stylized model of a firm with a single fixed project producing deterministic cash flows over a finite time interval, in which both entrepreneur and investors are risk neutral and the entrepreneur's human capital is certain, they show that there is a continuum of optimal debt contracts involving more or less rapid debt repayment paths. They also show that there is a unique optimal debt contract when the entrepreneur and investors have different discount rates. They argue that their framework provides a new foundation for a theory of corporate debt as well as a theory of debt maturity, and that their model "does not have room for equity per se" [pp 865].

In this paper we generalize the framework in Hart and Moore (1994) along several important dimensions: first, we introduce risky human capital and cash flows; second, we let the entrepreneur be risk averse; third, we consider an infinitely-lived firm with ongoing investment and consumption; and, fourth we add a limited liability or commitment constraint for investors. In this significantly more realistic and intricate framework we derive the entrepreneur's optimal financing, investment and consumption policy, and show how the firm's optimal financial contract can be implemented using replicating portfolios of standard securities.

A first obvious reason for considering this more involved framework is to explore how the Hart and Moore theory of debt based on the inalienability of human capital generalizes and how the introduction of risky human capital modifies the theory. But, more importantly, our framework reveals that Hart and Moore's focus on the notion of a firm's "debt capacity" is misleading. As it turns out, this is not the most relevant metric for the firm's optimal financial policy when human capital is risky. Rather, we show that the two key state variables summarizing the firm's financial policy are the firm's liquidity buffer or "financial slack" and the firm's hedging position or "risk management". That is, inalienability of *risky* human capital is not a foundation for a theory of debt capacity, but rather a foundation for a theory of *corporate liquidity and risk management*. There are hints of the relevance of corporate liquidity in Hart and Moore's discussion of their theory[1], however they do not emphasize the importance of this variable. Also, as a result of the absence of any risk in their framework they overlook the importance of the firm's hedging policy.

For convenience we introduce risk in the form of shocks to the capital stock, which affect the profitability of investment. Most importantly, these shocks induce risky inalienable human capital and introduce a stochastic dynamic participation constraint for the entrepreneur. That is, whether the entrepreneur is willing to stay with the firm now depends on the history of realized capital shocks. When there is a positive shock, the entrepreneur's human capital is higher and she must receive a greater promised compensation to be induced to stay. But the entrepreneur is averse to risk and has a preference for smooth consumption. These two opposing forces give rise to a novel dynamic optimal contracting problem between the infinitely-lived risk-averse entrepreneur and the fully diversified (or risk-neutral) investors.

A key step in our analysis is to show that the optimal long-term contracting problem between investors and the entrepreneur can be reduced to a recursive formulation with a single key state variable w, the ratio of the entrepreneur's *promised* certainty equivalent wealth under the contract and her human capital. The optimal recursive contract then specifies three state-contingent variables: *i*) the entrepreneur's consumption-capital ratio $c(w)$; *ii*) the firm's investment-capital ratio $i(w)$, and; *iii*) the firm's risk exposure $x(w)$ or hedging policy. This contract maximizes investors' payoff while providing insurance to the entrepreneur and retaining her. The optimal contract thus involves a particular form of the well-known tradeoff between risk sharing and incentives in a model of capital accumulation and limited commitment. Here the entrepreneur's dynamic participation constraint at each point in time is in effect her incentive constraint. She needs to be incentivized to stay rather than deploy her human capital elsewhere.

If the entrepreneur were able to alienate her human capital the optimal contract would simply provide her with a constant flow of consumption and shield her from any risk. Under this contract the firm's investment policy reduces to the familar Tobin's Q based policy. But

[1]On pages 864-865 they write: "There is some evidence that firms borrow more than they strictly need to cover the cost of their investment projects, in order to provide themselves with a "financial cushion." This fits in with our prediction in Proposition 2 about the nature of the slowest equilibrium repayment path; indeed, it is true of most paths."

under inalienable human capital the entrepreneur must be prevented from leaving, especially when accumulated capital is high. To retain the entrepreneur in these states of the world the optimal contract must promise her higher wealth and consumption when the capital stock is high, thus exposing her to risk.

Following the characterization of the optimal dynamic corporate policy $(c(w), i(w), x(w))$ we proceed with the implementation of this policy in terms of familiar standard dynamic financing structures. In particular, we show that the optimal contract can be implemented by delegating control over the firm to the entrepreneur in exchange for a credit line with an endogenously determined stochastic limit $\underline{S}$. The entrepreneur then maximizes her lifetime utility by optimally choosing her consumption-capital ratio $c(s)$, investment-capital ratio $\iota(s)$, and hedge-capital ratio $\phi(s)$ as a function of her savings s. In other words, the optimal contract under risky inalienable human capital can be implemented via a credit line combined with optimal cash management and dynamic hedging policies.

The optimal contract provides the entrepreneur with a flat consumption stream as long as the capital stock does not grow too large. When the capital stock increases as a result of investment or positive shocks to the point where the entrepreneur's participation constraint may be violated the contract provides a higher consumption stream to the entrepreneur. Given that the entrepreneur's consumption and wealth are positively correlated with the capital stock under the optimal contract the firm will generally underinvest relative to the benchmark of fully alienable human capital, as long as investors can perfectly commit to an optimal stochastic credit-line limit $\underline{S}$ (what we refer to as the one-sided commitment problem).

In the two-sided commitment problem, where a limited liability constraint for investors must also hold, we obtain further striking results. The firm may now *overinvest* and the entrepreneur *overconsume* (compared with the first-best benchmark). The intuition is as follows. In order to make sure that investors do not have incentives to default on their promised future utility for the entrepreneur, the entrepreneur's scaled promised wealth w cannot be too high otherwise the investors will end up with negative valuations for the firm. As a result, the entrepreneur needs to substantially increase investment and consumption in order to satisfy the investors' participation constraint.

Related literature. Our paper provides foundations for a dynamic theory of liquidity and risk management based on risky inalienable human capital. As such it is obviously related to the early important contributions on corporate risk management by Stulz (1984), Smith and Stulz (1985) and Froot, Scharfstein, and Stein (1993). Unlike our setup, they consider a static framework with exogenously given financial frictions to show how corporate cash and risk management can create value by relaxing these financial constraints.

Our paper is also evidently related to the corporate security design literature, which seeks to provide foundations for the existence of corporate financial constraints, and for the optimal external financing by corporations through debt or credit lines. This literature can be divided into three separate strands. The first approach provides foundations for external debt financing in a static optimal contracting framework with either asymmetric information and costly monitoring (Townsend, 1979, and Gale and Hellwig, 1985) or moral hazard (Innes, 1990, and Holmstrom and Tirole, 1997).

The second more dynamic optimal contracting formulation derives external debt and credit lines as optimal financial contracts in environments where not all cash flows generated by the firm are observable or verifiable (Bolton and Scharfstein, 1990, DeMarzo and Fishman, 2007, Biais, Mariotti, Plantin, and Rochet, 2007, DeMarzo and Sannikov, 2006, Biais, Mariotti, Rochet, 2010, and DeMarzo, Fishman, He and Wang, 2012; see Sannikov, 2012, for a recent survey of this literature).

The third approach which is closely related to the second provides foundations for debt financing based on the inalienability of human capital (Hart and Moore, 1994, 1998). Rampini and Viswanathan (2010, 2013) extend this framework to consider when corporate risk management add value. A key result in their framework is that hedging may not be an optimal policy for firms with limited capital that they can pledge as collateral. For such firms hedging demand, in effect, competes for limited collateral with investment demand. They show that for growth firms the return on investment may be so high that it crowds out hedging demand. Li, Whited, and Wu (2014) structurally estimate optimal contracting problems with limited commitment along the line of Rampini and Viswanathan (2013) providing empirical evidence in support of these class of models.

The latter two approaches are often grouped together because they yield closely related results and the formal frameworks are almost indistinguishable under the assumption of risk-neutral preferences for the entrepreneur and investors. However, as our analysis with risk-averse preferences for the entrepreneur makes clear, the two frameworks are different, with the models based on non-contractible cash flows imposing dynamic incentive constraints that restrict the set of incentive compatible financial contracts, while the models based on inalienable human capital only impose (dynamic) participation constraints for the entrepreneur. With the exception of Gale and Hellwig (1985) the corporate security design literature makes the simplifying assumption that the contracting parties are risk neutral. By allowing for risk-averse entrepreneurs, we not only generalize the results of this literature on the optimality of debt and credit lines, but we are able to account for the fundamental role of corporate savings and risk management (via futures or options or other commonly used derivatives), and also to provide micro-foundations for executive compensation contracts.

The most closely related papers to ours are by Ai and Li (2013) and Ai, Kiku, and Li (2013), who have independently considered a similar contracting framework to study how investment and managerial compensation vary with firm size and to derive an endogenous size distribution of firms that is consistent with Zipf's law. Our focus is somewhat different, as we seek to characterize the dynamics of optimal corporate liquidity and risk management.

Our paper also contributes to the macroeconomics literature that studies the implications of dynamic agency on firms' investment and financing decisions. Albuquerque and Hopenhayn (2004), Quadrini (2004) and Clementi and Hopenhayn (2006) study firms' financing and investment decisions under a limited commitment or inalienability of human capital assumption similar to ours. Lorenzoni and Walentin (2007) study Tobin's Q and investment under a similar limited enforcement assumption. Finally, Grochulski and Zhang (2011) consider a risk sharing problem under limited commitment.[2]

Our financial implementation of the optimal financial contract is also related to the portfolio choice literature featuring illiquid productive assets and under-diversified investors in an incomplete-markets setting. Building on Merton's intertemporal portfolio choice framework, Wang, Wang, and Yang (2012) study a risk-averse entrepreneur's optimal consumption-savings decision, portfolio choice, and capital accumulation when facing uninsurable id-

[2]Green (1987), Thomas and Worrall (1990), Marcet and Marimon (1992), Kehoe and Levine (1993) and Kocherlakota (1996) are important early contributions on optimal contracting under limited enforcement. See Ljunqvist and Sargent (2004) Part V for a textbook treatment on these class of models widely used in macro.

iosyncratic capital and productivity risks. Unlike Wang, Wang, and Yang (2012), our model features optimal liquidity and risk management policies that arise endogenously from an underlying financial contracting problem.

Our framework also provides a micro-foundation for the dynamic corporate savings models that take external financing costs as exogenously given. Hennessy and Whited (2005, 2007), Riddick and Whited (2009), and Eisfeldt and Muir (2014) study corporate investment and savings in a model with financial constraints. Bolton, Chen, and Wang (2011, 2013) study the optimal investment, asset sales, corporate savings, and risk management policies for a firm that faces external financing costs. It is remarkable that although these models are substantially simpler and more stylized the general results on the importance of corporate liquidity and risk management are broadly similar to those derived in our paper based on more primitive assumptions. Conceptually, our paper shows that to determine the dynamics of optimal corporate investment, a critical variable in addition to the marginal value of capital (marginal q) is the firm's marginal value of liquidity. Indeed, we establish that optimal investment is determined by the ratio of marginal q and the marginal value of liquidity, which reflects the tightness of external financing constraints.[3] Our model thus shares a similar focus on the marginal value of liquidity as Bolton, Chen, and Wang (2011, 2013) and Wang, Wang, and Yang (2012).

2 The model

We consider an optimal long-term contracting problem with limited commitment to participate between an infinitely-lived risk-neutral investor (*the principal*) and a financially constrained, infinitely-lived, risk-averse entrepreneur (*the agent*). The entrepreneur requires funding from the investor to finance a proprietary business idea for a growth venture that we represent as a production function and a capital accumulation process. We begin by describing the production technology and the entrepreneur and investor's preferences before formulating the dynamic optimal contracting problem between the two agents.

[3] Pinkowitz and Williamson (2004), Faulkender and Wang (2006), Pinkowitz, Stulz, andWilliamson (2006), Dittmar and Mahrt-Smith (2007), and Bolton, Schaller, and Wang (2014) empirically measure the marginal value of cash.

2.1 Capital Accumulation and Production Technology

Let I denote the gross investment by the entrepreneurial firm. We assume that the capital stock K accumulates as follows:

$$dK_t = (I_t - \delta K_t)dt + \sigma_K K_t dZ_t\,, \tag{1}$$

where Z is a standard Brownian motion, $\delta \geq 0$ is the expected rate of depreciation and σ is the volatility of a capital depreciation shock. This specification of shocks is used in equilibrium models including Cox, Ingersoll, and Ross (1985), Jones and Manuelli (2005), and Barro (2009), among others.

The firm's capital stock can be interpreted as either tangible capital (property, plant and equipment), firm-specific intangible capital (patents, know-how, brand value, and organizational capital), or any combination of these.

Production requires combining the entrepreneur's inalienable human capital with the firm's asset/production technology. When the two are united the firm's revenues are given by $A_t K_t$, where K_t is the firm's capital stock and $\{A_t;\, t \geq 0\}$ is a stochastic productivity shock. To keep the analysis simple, we model A_t as a two-state Markov regime-switching process, where $A_t \in \left\{A^L, A^H\right\}$ with $0 < A^L < A^H$, and λ_n is the transition intensity out of state $n = L$ or H to the other state.[4] In other words, given a current value of, say, $A_t = A^L$, the firm's productivity changes to A^H with probability $\lambda_L dt$ in the time interval $(t, t+dt)$. The productivity process $\{A_t;\, t \geq 0\}$ is observable to both the investor and entrepreneur, and is therefore contractible. The firm's output is given by $A_t K_t$.

Accumulating capital incurs both purchase and also adjustment costs. The firm's cash flows (after these capital costs) are given by:

$$Y_t = A_t K_t - I_t - G(I_t, K_t), \tag{2}$$

where the price of the investment good is normalized to unity and $G(I, K)$ is the standard adjustment cost function in the q-theory of investment of Hayashi (1982) and Abel and

[4] Piskorski and Tchistyi (2007) consider a model of mortgage design in which they use a Markov-switching process to describe interest rates. DeMarzo, Fishman, He, and Wang (2012) use a Markov-switching process to model the persistent productivity shock.

Eberly (1994). Importantly, Y can be negative which means that the investor would be financing investment I and associated adjustment costs G beyond the current revenue AK. We follow this literature and assume that the firm's adjustment cost $G(I,K)$ is homogeneous of degree one in I and K, so that $G(I,K)$ takes the following homogeneous form:

$$G(I,K) = g(i)K, \tag{3}$$

where $i = I/K$ denotes the firm's investment-capital ratio and $g(i)$ is an increasing and convex function. As Hayashi (1982) has first shown, with this homogeneity property Tobin's average and marginal q are equal under perfect capital markets.[5] However, as we will show, under limited commitment to participate an endogenous wedge between Tobin's average and marginal q will emerge in our model.[6] Note that (2) does not yet incorporate managerial compensation, which we will include later.

Hart and Moore (1994) is a simple version of our model without production. That is, our model boils down to the framework considered by Hart and Moore (1994), when we set: *i)* $\sigma_K = 0$ so that there are no shocks to the capital stock; *ii)* $\delta = 0$, so that the initial capital does not depreciate; *iii)* $I_t = 0$, so that there is no endogenous capital accumulation; and *iv)* $A_t = A > 0$ for all $t \geq 0$, so that there are no shocks to earnings. In other words, our framework adds to the basic Hart and Moore (1994) setup an endogenous capital accumulation process and shocks to both productivity and capital. Our goal is to explore the interactions of productivity shocks $\{A_t; t \geq 0\}$ and capital shocks $\{Z_t; t \geq 0\}$ on corporate investment/asset sales, financial slack, risk management, and managerial compensation, when the entrepreneur and investor contract under limited commitment to participate as in Hart and Moore (1994).

[5] Lucas and Prescott (1971) analyze dynamic investment decisions with convex adjustment costs, though they do not explicitly link their results to marginal or average q. Abel and Eberly (1994) extend Hayashi (1982) to a stochastic environment and a more general specification of adjustment costs.

[6] An endogenous wedge between Tobin's average and marginal q also arises in cash-based optimal financing and investment models such as Bolton, Chen, and Wang (2011) and optimal contracting models such as DeMarzo, Fishman, He, and Wang (2012).

2.2 Preferences

We further generalize the Hart and Moore (1994) setup by introducing risk aversion for the entrepreneur. Thus, the infinitely-lived risk-averse entrepreneur has a standard time-additive separable expected utility function over expected positive consumption flows $\{C_t; t \geq 0\}$ given by:

$$V_t = \mathbb{E}_t \left[\int_t^{\infty} \zeta e^{-\zeta(v-t)} U(C_v) dv \right] , \tag{4}$$

where $\zeta > 0$ is the entrepreneur's subjective discount rate, $U(C)$ is an increasing and concave function, and $\mathbb{E}_t [\,\cdot\,]$ is the time-t conditional expectation. We assume that the entrepreneur has constant relative risk aversion (CRRA) preferences and that $U(C)$ takes the standard iso-elastic constant-relative-risk-averse (CRRA) utility form:

$$U(C) = \frac{C^{1-\gamma}}{1-\gamma} , \tag{5}$$

where $\gamma > 0$ is the coefficient of relative risk aversion.[7] W can generalize our model with essentially no technical complication to separate the coefficient of relative risk aversion from the inverse of the elasticity of intertemporal substitution by using Epstein-Zin preferences.[8]

2.3 The Entrepreneur's Outside Option

The entrepreneur's human capital is inalienable and she can at any time leave the firm. When the entrepreneur exits she obtains an outside payoff of $\widehat{V}_n(K_t)$ (in utils) in state $n \in \{L, H\}$. In other words, $\widehat{V}_n(K_t)$ is the entrepreneur's outside value function, which in general is endogenous and depends on accumulated capital K_t and productivity A^n at the moment of exit as well as other features of the economic environment characterizing the entrepreneur's outside option. Given the endogenous outside option value (in utils) for the entrepreneur, the entrepreneur's participation constraint at each point in time t is therefore given by:

$$V_t \geq \widehat{V}_n(K_t) , \quad t \geq 0. \tag{6}$$

[7] Note that since we have normalized the value function with the constant ζ in (4), the utility flow in our model is $\zeta U(C)$ as in various dynamic contracting models.

[8] Detailed notes are available upon request. Intuitively, as Epstein-Zin preferences are homothetic, our model analysis will not increase the dimensionality of the optimization problem.

Our formulation of the entrepreneur's interim participation constraints can be interpreted in several ways. A first natural interpretation is that when she quits the entrepreneur absconds with a fraction $\alpha \in (0,1)$ of the firm's capital stock and find a new investor to finance her venture. The fraction of capital she makes off with may for example include all the know-how she has acquired running the firm, the firm's proprietary technology, and trade secrets. A second interpretation is that the entrepreneur puts her human capital to use in a competing firm where she can obtain a maximum payoff of $\widehat{V}_n(K_t)$. In this case, there is no diversion and the outside option value simply reflects the entrepreneur's human capital. A third interpretation is that the entrepreneur absconds a fraction of capital stock and then operates in autarky and hence forgoes inter temporal consumption-smoothing opportunities.[9] To intuitively exposit our analysis, we will use the first interpretation. We will also discuss the alternatives in Section 7.

We thus generalize the setup in Hart and Moore (1994) by introducing a risky outside option for a risk-averse entrepreneur in addition to substantially enriching the entrepreneur's production technology and providing a dynamic valuation framework in the spirit of q-theoretic framework. We show that the riskiness of the entrepreneur's inalienable human capital and risk aversion significantly enrich the optimal contract, yielding broad predictions that are consistent with observed dynamic corporate financial policies. Since the entrepreneur is assumed to be risk neutral in Hart and Moore (1994), their model features indeterminacy on the equilibrium repayment path. In our model, as we will show, the entrepreneur's risk aversion allows us to uniquely pin down the equilibrium repayment path.

2.4 The Contracting Problem

Without loss of generality we assume that the investor has all the bargaining power and that the contracting game begins at time 0 with the investor making a take-it-or-leave-it long-term contract offer to the entrepreneur, which specifies funding for an investment process $I = \{I_t; t \geq 0\}$, and a consumption allocation $C = \{C_t; t \geq 0\}$ for the entrepreneur, in return for the business income $\{Y_t; t \geq 0\}$. The investment and consumption processes can depend on the entire history of stochastic productivity $\{A_t; t \geq 0\}$, capital stock $\{K_t; t \geq 0\}$, and

[9] This interpretation is commonly used in the international macro literature. See Bulow and Rogoff (1989).

output $\{Y_t; t \geq 0\}$.

We consider two-sided limited-enforcement frictions where neither the investor nor the entrepreneur can fully commit to continuing under the contract in perpetuity. We assume that the investor is protected by limited liability and he can't commit to a long-term contract that yields negative net present value at any point in time. Specifically, at each point in time t, investors cannot commit to *ex post* negative net present value (NPV) projects, which implies the following constraints:

$$F_t \equiv \mathbb{E}_t \left[\int_t^{\infty} e^{-r(v-t)} (Y_v - C_v) dv \right] \geq 0 \,. \tag{7}$$

As we will show, this constraint plays a very important role for the optimal contract. For example, this constraint can generate over-investment in equilibrium. While surprising, it is intuitive as the investor also has incentive problems in this formulation. By over-investing, it relaxes the constraints for the investor to deviate from the optimal contract.

While with a two-sided limited-commitment friction, a long-term contract for the entrepreneur and the investor has the spirit of an optimal stopping game where both the entrepreneur and investors can exercise their perpetual American options at any time, in equilibrium, the contract will be set such that neither will exercise their options.

At the moment of contracting, time 0, the entrepreneur has a reservation utility level $\underline{V_0}$, so that the optimal contract must also satisfy the constraint:

$$V_0 \geq \underline{V_0} \,. \tag{8}$$

Without loss of generality, we require that $\underline{V_0} \geq \widehat{V}_n(K_0)$ where the value of A_0 is A^n with $n \in \{L, H\}$, for otherwise the entrepreneur would immediately walk away at time 0.

We assume that the output process Y is publicly observable and verifiable. In addition, we assume that the entrepreneur cannot privately save, as is standard in the literature on dynamic moral hazard (see Bolton and Dewatripont, 2005 chapter 10). Under these assumptions the only agency problem that arises in our setting is due to the entrepreneur's limited-commitment problem and the investors' limited liability constraint.

The investor's problem at time 0 is thus to choose dynamic investment I and consumption

C to maximize the time-0 discounted value of cash flows,

$$\max_{I,C} \quad \mathbb{E}_0 \left[\int_0^{\infty} e^{-rt}(Y_t - C_t)dt \right] , \tag{9}$$

subject to the capital accumulation process (1), the production function (2), the entrepreneur's limited-commitment constraints (6) at all t and investor's limited-liability condition (7) at all t, and the time-0 participation constraint (8).

Intuitively, the participation (8) constraint is always binding under the optimal contract. Otherwise, the investors can always increase their value by lowering the agent's consumption and still satisfying all other constraints. However, the entrepreneur's limited-commitment constraints (6) and the investors' limited-liability constraint (7) will often not bind as the investor dynamically trades off the benefits of providing the entrepreneur with risk-sharing/consumption smoothing and the costs of causing the investors' and the entrepreneur's limited commitment constraints to bind too often.

3 The Full-Commitment First-Best Benchmark

Before fully characterizing the optimal contract, we first analyze the optimal outcome under full commitments by both investors and the entrepreneur. Our contracting problem then reduces to the neoclassical setting of Hayashi (1982) with stochastic productivity. Intuitively, the risk-neutral investor simply buys off the entire venture from the risk-averse entrepreneur at time 0 for the reservation utility $\underline{V_0}$ and takes on all the output risk. The investor then maximizes the present discounted value of the venture's cash flows with respect to I.

Given the stationarity of the economic environment and the homogeneity of the production technology with respect to K, there is an optimal productivity-dependent investment-capital ratio $i_n = I/K$ in state $n \in \{L, H\}$ that maximizes the present value of the venture. The following proposition summarizes the main results under full commitment.

Proposition 1 *In each state $n \in \{L, H\}$, the firm's value $Q_n^{FB}(K)$ is proportional to its* *capital K, $Q_n^{FB}(K) = q_n^{FB} K$, where q_n^{FB} is Tobin's q in state n. For state H, q^H solves:*

$$(r+\delta)\, q_H^{FB} = \max_i \ \left(A^H - i - g(i)\right) + \lambda_H \left(q_L^{FB} - q_H^{FB}\right)\,, \tag{10}$$

and the maximand for (10), denoted by i_H^{FB}, is the first-best investment-capital ratio.

Because of homogeneity, return and present value relations hold for both the whole firm and also each unit of capital K. The first term on the right side of (10), $A^H - i - g(i)$, is the firm's cash flow per unit of time. And the second term, $\lambda_H \left(q_L^{FB} - q_H^{FB}\right)$, is the expected capital gains per unit of time, given by the product of probability λ_H per unit of time and the realized capital gains/losses $q_L^{FB} - q_H^{FB}$ conditional on the change of productivity from H to L. At optimality, the expected rate of return on capital is given by the sum of the discount rate r and the expected deprecation rate of capital δ, explaining the left side of (10). An essentially similar (and symmetric) valuation equation holds for q_L^{FB}.

Note that q_n^{FB} is the familiar Tobin average q, which under constant returns to scale is also the marginal value of capital, often referred to as marginal q, as Hayashi (1982) has shown in a deterministic setting. Adjustment costs create a wedge between the value of installed capital and newly purchased capital, so that that $q^{FB} \neq 1$ in general.

We can also express Tobin's q and optimal investment via the first-order condition (FOC):

$$q_n^{FB} = 1 + g'(i_n^{FB}), \quad n = L, H, \tag{11}$$

which states that the marginal q, is equal to the marginal cost of investing, $1+g'(i)$, evaluated at the optimum investment i_n^{FB}. By jointly solving (10) and (11) and similar two equations for state L, we obtain the values for q_n^{FB} and i_n^{FB}, where $n \in \{L, H\}$.

For the full-commitment case, the entrepreneur is perfectly insured and obtains a deterministic consumption stream that is independent of the firm's investment dynamics:

$$C_t = C_0\, e^{-(\zeta - r)t/\gamma}\,, \quad t \geq 0\,. \tag{12}$$

To the extent that the investor and entrepreneur have different discount rates, $\zeta \neq r$, the optimal contract will be structured so that they can trade consumption intertemporally with each other. Specifically, consumption changes exponentially at a rate $-(\zeta - r)/\gamma$ per unit of time, where $1/\gamma$ should be interpreted as the elasticity of intertemporal substitution, which is the inverse of the coefficient of relative risk aversion for standard CRRA utility functions. Thus, depending on the sign of $(\zeta - r)$ the entrepreneur's consumption may grow or decline deterministically over time. It is only when the investor and the entrepreneur are equally impatient ($\zeta = r$) that the entrepreneur's consumption is constant over time under the optimal full-commitment contract.

To complete the solution for the full-commitment case, we now explicitly solve the initial consumption C_0. First, we show that for a given level of the entrepreneur's utility V, we can calculate the corresponding certainty equivalent wealth (CEW) by inverting the expression $V(W) = U(bW)$ and obtain:

$$W = U^{-1}(V)/b, \tag{13}$$

where $U^{-1}(\,\cdot\,)$ is the inverse function of the CRRA utility (5) and b is a normalization constant given by

$$b \;=\; \zeta \left[\frac{1}{\gamma} - \frac{r}{\zeta}\left(\frac{1}{\gamma} - 1\right)\right]^{\frac{\gamma}{\gamma-1}} . \tag{14}$$

As a special case, when $\gamma = 1$, we have $b = \zeta e^{\frac{r-\zeta}{\zeta}}$. Because the entrepreneur's participation constraint (6) at time 0 will always bind due to the investor's optimality, the entrepreneur's reservation utility $\underline{V_0}$ implies that the initial CEW $\underline{W_0}$ satisfies $\underline{W_0} = U^{-1}(\underline{V_0})/b$. And the entrepreneur's initial consumption C_0 is proportional to $\underline{W_0}$, in that

$$C_0 = \chi \underline{W_0} = \left(\frac{\zeta}{b}\right)^{\frac{1}{\gamma}} ((1-\gamma)\underline{V_0})^{\frac{1}{1-\gamma}} , \tag{15}$$

where χ is the marginal propensity to consume (MPC) given by

$$\chi = b^{1-\frac{1}{\gamma}} \zeta^{\frac{1}{\gamma}} = r + \gamma^{-1} (\zeta - r) . \tag{16}$$

In the Appendix, we show that the entrepreneur's utility process, denoted by V_t^{FB}, under the first-best setting is then given by

$$V_t^{FB} = U(bW_t) = U(bC_t/\chi) \sim \underline{V_0}\, e^{-(\zeta-r)(1-\gamma)t/\gamma}\,, \tag{17}$$

where $U(\,\cdot\,)$ is given by (5). For the special case where $\zeta = r$, the entrepreneur's utility under the first-best case is time-invariant, in that $V_t^{FB} = U(C_0) = V_0$, as consumption is flat at all times. Because the entrepreneur's outside option value $\widehat{V}_n(K_t)$ growths stochastically with K, the entrepreneur's participation constraint will bind at some point, which will then lead to a renegotiation of the original first-best contract implying that the first-best contract is infeasible under limited-commitment constraints.

In summary, under perfect capital markets, the first-best investment-capital ratio i^{FB} depends on the current state $n \in \{L, H\}$ but is independent of capital shocks and the investor perfectly insures the entrepreneur's consumption smoothing preference. As we will show next, the entrepreneur's inability to fully commit to the venture indefinitely and the investors' limited liability significantly alters these conclusions.

4 Optimal Dynamic Contracting

The first-best outcome is not achievable when the entrepreneur or investors cannot commit to stay. The simple reason is that under the first-best optimal investment policy, the firm's capital stock grows (in expectation) over time and there will be cut-off values of $\overline{K}_t^H$ and $\overline{K}_t^L$ such that when $K_t > \overline{K}_t^H$ in state H or when $K_t > \overline{K}_t^L$ in state L, we will have $\widehat{V}_n(K_t) > V_t^{FB}$, where V_t^{FB} is given by (17). In this case, the first-best contract will not be honored by the entrepreneur as she would simply walk away at that point. To prevent such an outcome the investor writes a second-best contract where he commits to a consumption flow $\{C_t : t \geq 0\}$ for the entrepreneur such that $V_t \geq \widehat{V}_n(K_t)$ at all time t and in both H and L productivity states. Since $\widehat{V}_n(K_t)$ is a stochastic process, this second-best contract would inevitably expose the entrepreneur to consumption risk. Similar arguments apply to the friction induced by investors' limited liability condition. While it protects investors from losses ex post, limited liability limits the contracting space. As a result, there will be states of the world where K_t is such that the investors' value is negative.

In summary, our dynamic contracting problem involves a specific form of the classic agency tradeoff between risk-sharing provision to the agent and incentive mis-alignment induced by the entrepreneur's and/or investors' limited commitment problems. An important difference, however, from the standard dynamic moral hazard problem is that the entrepreneur and investor's dynamic participation constraint often will not bind. The reason is that if the contract under two-sided commitment were to always hold the entrepreneur down to her participation constraint or up to investor's participation constraint then the entrepreneur's promised consumption would be inefficiently volatile.

4.1 Formulating the optimal contracting problem

The second-best dynamic contracting problem involves making contingent investment $\{I_t; t \geq 0\}$ and consumption promises $\{C_t; t \geq 0\}$ to the entrepreneur. Generally, the contracting problem solution is history-dependent. Importantly, we can summarize the history dependence by using the entrepreneur's promised utility V, as in DeMarzo and Sannikov (2006) among others.

The optimal contract requires that the sum of utility flow $\zeta U(C_{t-}))dt$ and the change of the agent's promised utility dV_t has an expected value $\zeta V_t dt$ in that

$$\mathbb{E}_{t-}\left[\zeta U(C_{t-}))dt + dV_t\right] = \zeta V_{t-}dt\,, \tag{18}$$

To provide some intuitive reasoning for (18), we construct a stochastic process $\{\widehat{U}_t, t \geq 0\}$ as follows:

$$\widehat{U}_t = \int_0^t e^{-\zeta v}\zeta U(C_v)dv + e^{-\zeta t}V_t = \mathbb{E}_t\left[\int_0^\infty \zeta e^{-\zeta v}U(C_v)dv\right]. \tag{19}$$

Under technical integrability conditions, we know that $\{\widehat{U}_t; t \geq 0\}$ is a martingale in that $\mathbb{E}_t[\widehat{U}_s] = \widehat{U}_t$ for all s and t such that $s > t$. Applying Ito's formula to the marginal process $\widehat{U}$ given in (19) and using the property that a martingale's drift is zero, we obtain (18).

Intuitively, delivering a marginal unit of consumption to the entrepreneur lowers his promised utility V by reducing its drift ζV_{t-} by $\zeta U(C_{t-})$, and hence we have the following equivalent representation of (18):

$$\mathbb{E}_{t-}\left[dV_t\right] = \zeta\left(V_{t-} - U(C_{t-})\right)dt. \tag{20}$$

Recall that there are two shocks, the capital shock (via the Brownian motion Z) and the productivity shock (via the two-state Markov chain.) So we may write down the stochastic differential equation (SDE) for dV implied by (18) as the sum of (i) the expected change (i.e., drift) term $\mathbb{E}_{t-}\left[dV_t\right]$, (ii) a martingale term driven by the Brownian motion Z, and (iii) a martingale term driven by the productivity shock driven by the Markov chain.

Letting N_t denote the cumulative number of productivity changes up to time t. For notational simplicity, suppose that the current productivity at time $t-$ is H. We may write the dynamics of the entrepreneur's promised utility process V as follows:

$$dV_t = \zeta(V_{t-} - U(C_{t-}))dt + x_{t-}V_{t-}dZ_t + \Gamma_H(V_{t-}, A^H)(dN_t - \lambda_H dt)\,, \tag{21}$$

where $\{x_t; t \geq 0\}$ controls the diffusion volatility of the entrepreneur's promised utility V, and $\Gamma_H(V_{t-}, A^H)$ controls the endogenous adjustment of promised utility V conditional on the change of productivity from A^H to A^L. Intuitively, the first term on the right side of (21) is the expected change of dV_t as implied by (18), the second term is the unexpected change due to capital shock Z, and the last term captures the mean-zero unexpected component of dV_t due to the change of productivity. As λ_H is the probability per unit of time for productivity to switch from A^H to A^L, the expected value of $(dN_t - \lambda_H dt)$ is zero.

By using the entrepreneur's promised utility process V, we make the investor's dynamic contracting problem Markovian with the following three state variables: (1) the entrepreneur's promised utility V, (2) the venture's capital stock K, and (3) the state of productivity $n \in \{L, H\}$. Let $F(K, V, A^n)$ denote the investor's value function.

The contract specifies dynamic investment I, consumption C, as well as risk exposure x and the adjustment of promised utility Γ_n to solve the following optimization problem,

$$F(K_t, V_t, A^n) = \max_{C, I, x, \Gamma_n} \quad \mathbb{E}_t\left[\int_t^\infty e^{-r(v-t)}(Y_v - C_v)dv\right]\,, \tag{22}$$

subject to the entrepreneurs' limited commitment constraint (6) and the investors' limited-liability condition (7) as well as the entrepreneur's initial participation constraint (8).

Next, we characterize the investor's optimization problem in the interior region and then we characterize the boundary conditions.

The interior region. For expositional simplicity, suppose that the current state is H. Then, the following Hamilton-Jacobi-Bellman (HJB) equation holds:

$$\begin{aligned} rF(K,V,A^H) \;=\; \max_{C,I,x,\Gamma_H} & \Big\{ Y - C + (I-\delta K)F_K + \frac{\sigma_K^2 K^2}{2}F_{KK} + [\zeta(V-U(C)) - \lambda_H \Gamma_H]F_V \\ & + \frac{(xV)^2}{2}F_{VV} + \sigma_K xKVF_{VK} + \lambda_H[F(K,V+\Gamma_H,A^L) - F(K,V,A^H)] \Big\} \end{aligned} \tag{23}$$

The right side of (23) gives the expected change of the investor's value function $F(K,V,A^H)$. The first term is the venture's flow profit $Y-C$ to the investor, which can be negative. In this case, the investor is financing operating losses. The second term reflects the expected change of the investor's value $F(K,V,A^H)$ resulting from the expected (net) capital accumulation $(I-\delta K)$ and the third term represents the expected change in the investor's value resulting from the volatility of the capital shock. These first three terms are the ones in standard resource allocation problem (between consumption and investment).

The fourth and fifth terms in turn reflect the change in investor's value from the drift and volatility of the entrepreneur's promised utility V. And the sixth term captures how the investor's value is affected by the (perfect) correlation between K and V. Finally, the last term captures the effect of the persistent productivity shock on the value function. Importantly, in addition to the direct effect on the investor's value F, the productivity switch from A^H to A^L also has an indirect effect on the investor's value F due to the endogenous adjustment of the entrepreneur's promised utility from V to $V+\Gamma_H$.

As the investor earns the rate of return r at all times, the sum of all terms on the right side of (23) must equal to $rF(K,V,A^H)$ which is on the left side of (23). In state $n \in \{L,H\}$, the first-order conditions (FOCs) with respect to C, I, and x are:

$$\zeta U'(C^*) = -\frac{1}{F_V(K,V,A^H)}, \tag{24}$$

$$F_K(K,V,A^H) = 1 + G_I(I^*,K), \tag{25}$$

$$x^* = -\frac{\sigma_K K F_{VK}}{V F_{VV}(K,V,A^H)}, \tag{26}$$

The FOC (24) characterizes the entrepreneur's optimal consumption C^*, which states that the entrepreneur's marginal utility of consumption $\zeta U'(C^*)$ equals $-1/F_V$, which is positive as $F_V < 0$. Note that we need multiply $-F_V$ with the entrepreneur's marginal utility $\zeta U'(C)$ to calculate the investor's marginal benefit of increasing consumption. At optimality, this marginal benefit $-F_V \zeta U'(C)$ has to equal to unity, which is the risk-neutral investor's marginal cost of providing a unit of consumption.

Second, (25) characterizes the investors' optimal investment decision. Investment optimality implies that marginal benefit of investing to the investor, $F_K(K,V,A^n)$, must equal to $1 + G_I(I,K)$, the marginal cost of investing. Unlike the standard q-theoretic investment models, our model sets the marginal condition from the investor's perspective. Finally, (26) characterizes the optimal exposure of the entrepreneur's promised utility V to the shock of capital accumulation. As there is only one exogenous diffusion shock in the model, V and K are locally perfectly correlated. As we show later, x is closely tied to the firm's optimal risk management policy.

We now turn to the choices of Γ_H, the discrete change of the entrepreneur's promised utility contingent on the change of the productivity from H to L. Intuitively, whenever feasible, the optimal contract equates the investors' marginal cost of delivering compensation just before and after the productivity changes in that,

$$F_V(K, V + \Gamma_H^*, A^L) = F_V(K,V,A^H), \tag{27}$$

which is the FOC with respect to Γ_H implied by (23). Note that the second-order condition (SOC) is given by $F_{VV}(K, V + \Gamma_H^*, A^L) < 0$ which implies that F is concave in V at Γ_H^*.

However, (27) only holds when neither the entrepreneur's limited-commitment constraint nor the investor's limited-liability constraint binds. When either constraint binds, we will have inequalities rather than equalities for the FOC with respect to Γ_H. We will later return to the corner-solution case with more detailed discussions.

Next we turn to the boundary conditions where either the entrepreneur's limited-commitment constraint or the investors' limited liability constraint binds.

The boundary conditions. Consider the two-sided commitment problem. First, we consider the endogenous right boundary condition which arises from the investor's limited-liability considerations. Let $\overline{V}_n(K)$ denote the entrepreneur's time-t state-n promised utility such that the investor's limited-liability condition is met and hence will not voluntarily walk away. That is, in state n, $\overline{V}_n(K)$ satisfies

$$F(K, \overline{V}_n(K), A^n) = 0\,, \tag{28}$$

which gives the upper endogenous boundary condition.

Now, we turn to the left boundary condition that arises from the entrepreneur's dynamic participation constraint. At that endogenously determined boundary, the entrepreneur is indifferent between staying within the long-term relationship with the investor and walking away with the outside option.

As the entrepreneur has an outside option to divert a fixed fraction α $(0 < \alpha < 1)$ of the firm's capital and set up a new firm under an optimal long-term contract afresh with no liability, as sketched out in Section 2, we have an endogenous lower boundary for the entrepreneur's promised utility V. What is the entrepreneur's value function by following the deviation (diversion) strategy? The entrepreneur evaluates his consumption under a new optimal contract (with essentially the same term) but with zero liability (i.e. $F = 0$) and only α fraction of the firm's current capital stock. Therefore, the entrepreneur's outside option value $\widehat{V}_n(K_t)$ in state n should be given by $\overline{V}_n(\,\cdot\,)$, the entrepreneur's value function under zero liability, but only with αK_t, in that

$$\widehat{V}_n(K_t) = \overline{V}_n(\alpha K_t)\,, \tag{29}$$

where $\overline{V}_n(\,\cdot\,)$ is given by (60).

In summary, with the entrepreneur's limited commitment and the investor's limited liability, the entrepreneur's utility V_t must satisfy

$$\overline{V}_n(\alpha K_t) \leq V_t \leq \overline{V}_n(K_t). \tag{30}$$

The HJB equation (23), the FOCs (24), (25), (26) and (27) as well as boundary conditions jointly characterize the solution to the second-best optimal contract.

4.2 The Entrepreneurs' Promised Certainty Equivalent Wealth W

How do we link the entrepreneur's promised utility V, the key state variable characterizing the optimal compensation to variables that are empirically measurable? As we will show, the entrepreneur's promised (certainty-equivalent) wealth W can be naturally mapped to the firm's liquidity holding. Before delving into financial implementation, we first summarize the transformation from V to W.

As in the full-commitment first-best benchmark case, we may equivalently reformulate the investor's value function $F(K, V, A^n)$ to one expressed in terms of K and W, which we denote by $P(K, W, A^n)$ as follows,

$$P(K, W, A^n) \equiv F(K, V, A^n) = F(K, U(bW), A^n)\,, \quad n \in \{L,\ H\}, \tag{31}$$

where $V(W) = U(bW)$ links the entrepreneur's utility V with the promised wealth W.

By applying the Ito's formula to $P(K, W, A^n)$ and using (31), we obtain an HJB equation for $F(K, V, A^n)$ and also correspondingly FOCs for C, I, x, and the endogenous adjustments of the entrepreneur's promised certainty equivalent wealth, Ψ_H and Ψ_L, as the state of productivity switches from H to L, and from L to H, respectively.

By exploiting our model's homogeneity property, we may transform our PDE formulation for $P(K, W, A)$ into analytically tractable coupled ODE formulation for $p(w, A^H)$ and $p(w, A^L)$, where the entrepreneur's promised certainty equivalent wealth $P(K, W, A)$ is homogeneous of degree one in K and W, in that

$$P(K, W, A^n) = p_n(w)K\,, n \in \{L, H\}, \tag{32}$$

where $w = W/K$ is the entrepreneur's W scaled by the firm's capital stock K, and $p_n(w)$ is the scaled investors' value in state $n \in \{L, H\}$. The technical details for transformations are provided in the Appendix.

5 Implementation: Liquidity and Risk Management

Having characterized the optimal contract via the entrepreneur's promised wealth W, we next implement the model's predictions via commonly used financial instruments by posing an entrepreneur's optimization problem in an economic environment with these standard securities. We emphasize that the original contracting problem yields identical outcomes as the entrepreneur's optimization problem that we will introduce next. Importantly, the entrepreneur's limited commitments and/or the investor's limited liability naturally give rise to the entrepreneur's corporate risk management and liquidity management, implemented via standard liquidity and derivative (e.g., futures) securities.[10]

Liquidity management. We first describe the entrepreneur's liquidity management opportunity. We endow the entrepreneur with a bank/credit account. Let S_t denote this account's time-t balance and naturally $S_t < 0$ corresponds to the case where the entrepreneur is using the credit line from the bank. At each time t with productivity A^n, the entrepreneur can borrow at the risk-free rate r up to a maximal value of $\overline{D}_n(K_t)$, which we refer to as the *endogenous* debt capacity in state n. We show that $\overline{D}_n(K_t)$ is determined by the primal contracting problem and is given by

$$\overline{D}_n(K) = P(K, \underline{W_n}, A^n)\,, \qquad n \in \{L, H\}, \tag{33}$$

where $P(K, \underline{W_n}, A^n)$ is the investors' value when the entrepreneur's limited-commitment constraint binds, i.e. when $W_n = \underline{W_n}$. Intuitively, provided that the entrepreneur does not walk away from the contract, the entrepreneur's debt is then risk free and hence can be financed at the risk-free rate.

[10]It is well known that implementation is not unique. We choose an intuitive one and will discuss alternative ways of implementing the dynamic optimal contract.

However, liquidity management via a risk-free savings/credit account is not state contingent and thus has limited abilities for the entrepreneur to manage various risks. To replicate the optimal contracting outcome, we need additional state-contingent instruments to allow the entrepreneur to optimally manage the risk exposure.

Risk management against capital shocks. One way for the entrepreneur to manage the capital risk Z is to use a standard risk management instrument such as futures.[11] Consider the futures contract written on capital shock Z. Because investors are risk neutral, there is no premium to entering futures contract whose payoffs have zero mean. Suppose that a unit of long position in futures gives the holder an exposure of $\sigma_K dZ_t$. With a size of $\boldsymbol{\phi}_t K_t$ in the underlying futures, the entrepreneur's total exposure is then $\boldsymbol{\phi}_t K_t \sigma_K dZ_t$. As the profits/losses of the futures position are only subject to diffusion shocks and instantaneously credited/debited from the entrepreneur's bank account, there is no default risk.

Insurance against productivity shocks. Finally, we propose an insurance contract that allows the entrepreneur to hedge the stochastic change of the productivity state. Suppose that the current state is H. An investor with a long position in the insurance contract pays an insurance premium at the rate of λ_H but will receive a unit of payoff from the counterparty if and only if the productivity state switches from H to L. As the investor is risk neutral, the actuarially fair premium per unit of time for this insurance is indeed λ_H.

Let $\pi_H(S, A^H)K$ denote the entrepreneur's demand for this insurance contract in state H. The entrepreneur thus pays a total insurance premium $\pi_H(S, A^H)K\lambda_H$ per unit of time and receives a lump-sum payment $\pi_H(S, A^H)K$ if and only if the state switches from H to L, i.e. when $dN_t = 1$, but zero, otherwise. Therefore, the total stochastic exposure of this insurance contract is $\pi_H(S, A^H)K_t(dN_t - \lambda_H dt)$ where $dN_t \in \{1, 0\}$.

As the entrepreneur is risk averse, via the standard risk-sharing argument, we know that the demand for both futures and insurance contracts are generally not zero. We will later extend the model to allow for risk premium.

[11] Bolton, Chen, and Wang (2011) analyze the optimal corporate risk management for a financially constrained firm. In that model, they also analyze the dynamic futures trading strategies but their model is not a dynamic contracting framework.

Liquidity dynamics. After having presented the three financial instruments for our implementation, we now write down the dynamic evolution of the entrepreneur's savings balance, denoted by S_t. In state H, S_t evolves as follows,

$$dS_t = (rS_t + Y_t - C_t)dt + \boldsymbol{\phi}_t K_t \sigma_K dZ_t + \pi_H(S, A^H)K_t(dN_t - \lambda_H dt)\,, \tag{34}$$

as long as the following credit constraint (35) is satisfied:

$$S_t \geq -\overline{D}_H(K_t), \tag{35}$$

where $\overline{D}_H(K_t) > 0$ is given by (33).

The first term in (34), $rS + Y - C$, is given by the sum of the interest income rS and the investors' business income $Y - C$. In the absence of risk management and insurance, $rS + Y - C$ is simply the rate at which the entrepreneur saves or draws on the line of credit at the risk-free rate r. The second term $\boldsymbol{\phi}_t K_t \sigma_K dZ_t$ in (34) describes the effect of hedging the capital shock Z via the futures position $\boldsymbol{\phi} K$, and the last term $\pi_H(S, A^H)K_t(dN_t - \lambda_H dt)$ captures the effect of the insurance contract against productivity changes.

The Entrepreneur's Optimization Problem. We now summarize the implementation problem. The entrepreneur optimally chooses consumption C, investment I, (scaled) futures position $\boldsymbol{\phi}$ and scaled insurance demand, π_H and π_L, to maximize utility given by (4)-(5) subject to the liquidity accumulation dynamics (34) and the endogenous borrowing limit (35), where $\overline{D}_n(K)$ is specified in (33).

Guided by economic insights under the full-commitment case, we express the entrepreneur's value function $J(K, S, A^n)$ as follows,

$$J(K, S, A^n) = \frac{(bM(K, S, A^n))^{1-\gamma}}{1-\gamma}\,, \quad n = L, H\,. \tag{36}$$

Here, $M(K, S, A^n)$ can be interpreted as the entrepreneur's certainty equivalent wealth and the normalization constant b is given by (14). In the Appendix, we provide the details on how to characterize $M(K, S, A^n)$ via a PDE with corresponding boundary conditions.

To simplify the exposition of the key economic mechanism in our model, we next analyze the case with capital (diffusion) risk only, which is a special case with $A^L = A^H = A$.

6 The Diffusion-Only Case

In this section, we analyze the diffusion-only case by first summarizing the model's solution and then using the solution to analyze the model's main implications and results. We mostly focus on analyzing the implementation problem.

6.1 Solution

By exploiting our model's homogeneity property, we may transform our PDE formulation into analytically tractable ODE formulation. Specifically, we show that the entrepreneur's certainty equivalent wealth $M(K, S)$ is homogeneous of degree one in K and S, in that

$$M(K, S) = m(s)K \,, \tag{37}$$

where $s = S/K$ is the entrepreneur's savings S scaled by the firm's capital stock K, and $m(s)$ is the scaled promised (certainty equivalent) wealth.[12]

The dynamic of scaled liquidity s**.** Given the consumption-capital ratio $\mathfrak{c}(s)$, the investment-capital ratio $\mathfrak{i}(s)$, and the hedge ratio $\boldsymbol{\phi}(s)$, in the interior region, we have

$$ds_t = \mu^s(s_t)dt + \sigma^s(s_t)dZ_t \,, \tag{38}$$

where the drift and volatility processes $\mu^s(\,\cdot\,)$ and $\sigma^s(\,\cdot\,)$ for s are given by

$$\mu^s(s) \;=\; (A - \mathfrak{i}(s) - g(\mathfrak{i}(s)) - \mathfrak{c}(s)) + (r + \delta - \mathfrak{i}(s))s - \sigma_K \sigma^s(s), \tag{39}$$

$$\sigma^s(s) \;=\; (\boldsymbol{\phi}(s) - s)\sigma_K \,. \tag{40}$$

[12] Wang, Wang, and Yang (2012) solve an entrepreneur's optimal consumption-savings, business investment, and portfolio choice problem with endogenous entry and exit decisions. By exploiting homogeneity, they derive the optimal investment policy in a q-theoretic context with incomplete markets. In our model, we optimally implement the solution of the optimal contacting problem.

The one-sided limited-commitment case. The following proposition summarizes the solution for the case with only the entrepreneur's limited-commitment problem.

Proposition 2 *In the region $s > \underline{s}$, the entrepreneur's scaled promised wealth $m(s)$ solves:*

$$
\begin{aligned}
0 \;=\; & \max_{\mathfrak{i}(s)} \frac{m(s)}{1-\gamma}\left[\gamma\chi(m'(s))^{\frac{\gamma-1}{\gamma}}-\zeta\right]-\delta m(s)+\left[(r+\delta)s+A\right]m'(s) \\
& +\mathfrak{i}(s)(m(s)-(s+1)m'(s))-g(\mathfrak{i}(s))m'(s)-\frac{\gamma\sigma_K^2}{2}\frac{m(s)^2m''(s)}{m(s)m''(s)-\gamma m'(s)^2}\,, \qquad (41)
\end{aligned}
$$

subject to the following boundary conditions:

$$
\lim_{s\to\infty} m(s) \;=\; q^{FB}+s\,, \qquad (42)
$$

$$
m(\underline{s}) \;=\; \alpha m(0)\,, \qquad (43)
$$

$$
\lim_{s\to\underline{s}} \sigma^s(s) \;=\; 0 \quad \textit{and} \quad \lim_{s\to\underline{s}} \mu^s(s)\geq 0\,. \qquad (44)
$$

The non-linear ODE given by (41) characterizes the entrepreneur's scaled promised wealth $m(s)$ in the interior region $s > \underline{s}$. We now turn to boundary conditions. First, as the entrepreneur's savings s approaches infinity ($s \to \infty$), the entrepreneur's self insurance is sufficient to achieve the first-best resource allocation outcome, and hence $m(s)$ approaches $q^{FB} + s$, the sum of the Tobin's q under the first-best benchmark and the value of liquidity s. In this case, the marginal value of liquidity is simply unity as a financially unconstrained entrepreneur does not pay a premium for liquid assets, and the entrepreneur values per unit of capital K at its first-best maximal value q^{FB}.

Next, we turn to the endogenous boundary $\underline{s}$. Intuitively, at this boundary $\underline{s}$, the entrepreneur's promised wealth $m(s)$ equals $\alpha m(0)$, as the entrepreneur's certainty-equivalent wealth is $\alpha m(0)$ per unit of capital by following the deviation strategy of diverting α fraction of capital stock with zero liability. Additionally, how do we ensure that the entrepreneur does not default on debt as s approaches $\underline{s}$? We need to require that the volatility of s evaluated at $\underline{s}$ must be zero, as stated in (44), and moreover, the drift $\mu^s(\underline{s})$ should be weakly positive to ensure that the constraint $s \geq \underline{s}$ is satisfied at all times. These two conditions can be understood via reasoning by contradiction. First, consider the zero volatility condition at $s = \underline{s}$. Suppose $\lim_{s\to\underline{s}} \sigma^s(s) \neq 0$, then for values of s that are very close to the boundary

$\underline{s}$, the possibility of crossing the boundary $\underline{s}$ is strictly positive (as the volatility effect dominates the drift effect in diffusion) violating the constraint $s \geq \underline{s}$. Additionally, we also have to rule out a deterministic violation of $s \geq \underline{s}$, which will occur with a negative drift $\mu^s(\,\cdot\,)$ at $\underline{s}$ even when the volatility at $\underline{s}$ is zero. Thus, we also need $\mu^s(\underline{s}) \geq 0$.

The two-sided limited-commitment case. What if the investor also faces the limited liability constraint? For this two-sided limited commitment problem, we simply need to modify conditions at the the upper boundary in Proposition 2. First, we note that the investor's limited liability implies that the upper boundary is $s = 0$ rather than the natural limiting boundary $s \to \infty$ for the one-sided limited-commitment case. We thus replace condition (42) with the following conditions at the new upper boundary $s = 0$:

$$\lim_{s \to 0} \sigma^s(s) = 0 \quad \text{and} \quad \lim_{s \to 0} \mu^s(s) \leq 0\,. \tag{45}$$

The arguments for (45) are essentially the same as those we have laid out earlier for the lower boundary $\underline{s}$. Intuitively, at the upper boundary $s = 0$, the volatility $\sigma^s(\,\cdot\,)$ has to be zero and the drift needs to be weakly negative to pull s to the interior ensuring that s will not violate the constraint $s \leq 0$.

6.2 Parameter choices and calibration

Our diffusion model is parsimonious with only eight parameters. Three parameters essential for the contracting tradeoff between risk sharing and limited enforcement are risk aversion γ, volatility σ_K, and the diversion parameter α. Other five parameters (e.g., the risk-free rate r, the entrepreneur's discount rate ζ, depreciation rate δ, adjustment cost θ, and productivity A) are also critical to make the model fully dynamic and operational but in a simplest setting. We choose sensible parameter values to highlight the model's mechanism and main insights. Also, we provide a first-pass assessment on the quantitative importance of limited-commitment and limited-liability frictions.

We choose the widely used value for the coefficient of relative risk aversion, $\gamma = 2$. The annual risk-free interest rate is $r = 5\%$. The entrepreneur's annual subjective discount rate is set to equal to the risk-free rate, $\zeta = r = 5\%$. Under the first-best setting, consumption then should be constant at all times.

On the real investment side, we rely on the findings of Eberly, Rebelo, and Vincent (2009) who provide empirical evidence in support of Hayashi (1982). Following their work, we set the annual productivity $A = 20\%$ and the annual volatility of productivity shocks $\sigma = 20\%$.

While our model is equally tractable for any homogeneous adjustment cost function $g(i)$, we choose the following quadratic adjustment cost function for illustrational simplicity,

$$g(i) = \frac{\theta i^2}{2}, \tag{46}$$

where the parameter θ measures the degree of the adjustment cost.[13] A higher value θ implies a more costly adjustment process. For this case, we have explicit formulas for Tobin's q and the optimal investment-capital ratio i in the first-best MM benchmark:

$$q^{FB} = 1 + \theta i^{FB}, \text{ and } i^{FB} = r + \delta - \sqrt{(r+\delta)^2 - 2\frac{A-(r+\delta)}{\theta}}. \tag{47}$$

Fitting the first-best values of q^{FB} and i^{FB} to the sample averages, we set the adjustment cost parameter $\theta = 2$ and the (expected) annual depreciation rate for capital stock as $\delta = 12.5\%$. These baseline parameters imply $q^{FB} = 1.2$ and the annual investment-capital ratio $i^{FB} = 0.1$. We choose the fraction of capital stock that the entrepreneur may divert, α to be 0.4, broadly in line with empirical estimates.[14] The parameter values for our baseline case are summarized in Table 1. Note that all parameter values are annualized when applicable.

[13] Jermann (1998) proposes an isoelastic nomothetic adjustment cost and argue that the curvature of the adjustment cost is critical in equilibrium models.

[14] See Li, Whited, and Wu (2014) for the empirical estimates of α. The averages are 1.2 for Tobin's q and 0.1 for the investment-capital ratio, respectively, for the sample used by Eberly, Rebelo, and Vincent (2009). The imputed value for the adjustment cost parameter θ is 2 broadly in the range of estimates used in the literature. See Whited (1992), Hall (2004), Riddick and Whited (2009), and Eberly, Rebelo, and Vincent (2009).

Table 1: Summary of Parameters

This table summarizes the parameter values for the baseline model with $A^H = A^L$.

Parameters	Symbol	Value
Risk-free rate	r	5%
The entrepreneur's discount rate	ζ	5%
The entrepreneur's relative risk Aversion	γ	2
Capital depreciation rate	δ	12.5%
Volatility of capital depreciation shock	σ_K	20%
Quadratic adjustment cost parameter	θ	2
Firm's productivity	A	20%
Diversion parameter	α	40%

6.3 Promised Wealth W and Financial Slack S

As we have noted, the contracting problem and the implementation formulation are equivalent.[15] The liquidity variables in the two formulations are related to each other as follows:

$$s = -p(w) \text{ and } w = m(s)\,, \tag{48}$$

where $p(w)$ is the scaled investors' value as a function of promised scaled wealth w in the contracting problem and $m(s)$ is the entrepreneur's scaled certainty equivalent wealth as a function of scaled liquidity s. Importantly, (48) implies that the composition of $-p$ and m, denoted by $-p \circ m$, yields the identity function, i.e. $-p(m(s)) = s$.

Scaled promised wealth w and scaled investors' value $p(w)$. Figure 1 plots the investor's scaled value $p\,(w)$ and the sensitivity $p'\,(w) = P_W$ in Panels A and B, respectively. Note that $p(w)$ is decreasing in w for both the one-sided and two-sided limited-commitment cases. Intuitively, the higher the entrepreneur's promised certainty equivalent wealth w, the lower the investors' value $p(w)$. Moreover, as w increases, the entrepreneur becomes less constrained and the marginal value $p'(w)$ decreases.

[15] In the Appendix , we show that the ODE for $p(w)$ and the ODE (41) for $m(s)$ are equivalent. Of course, additionally, the boundary conditions and policy rules for the two formulations are also matched.

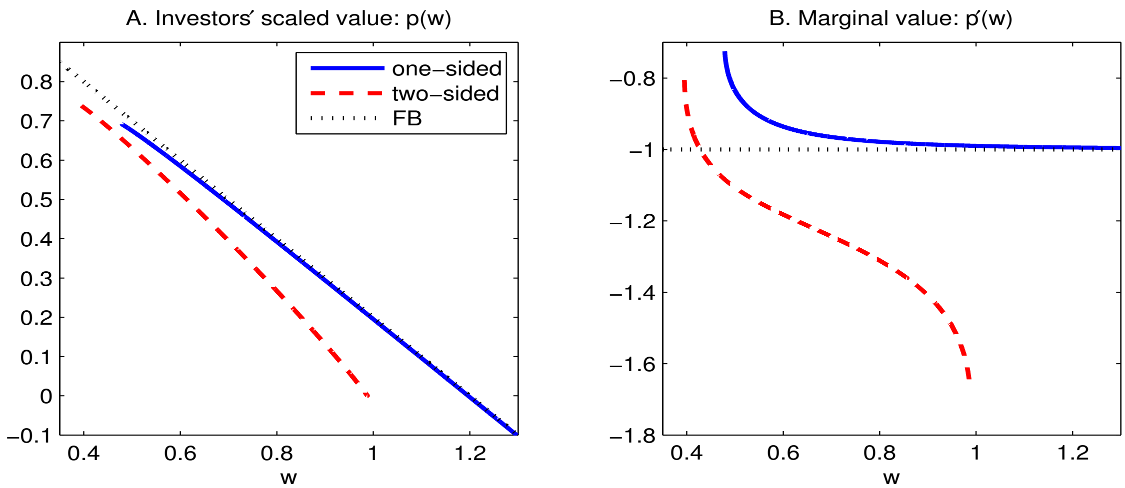


Figure 1: **Investors' scaled value $p(w)$ and the investors' marginal value of w, $p'(w)$, as functions of the entrepreneur's scaled promised wealth w.** The solid and dashed lines correspond to the one-sided and two-sided limited-commitment cases, respectively. For both cases, $p(w)$ is decreasing and concave in w. For the one-sided case, $w \geq \underline{w} = 0.479$. For the two-sided case, $0.4 = \underline{w} \leq w \leq \overline{w} = 0.988$. The dotted line depicts the first-best MM results: $p(w) = q^{FB} - w$ and the sensitivity $p'(w) = -1$.

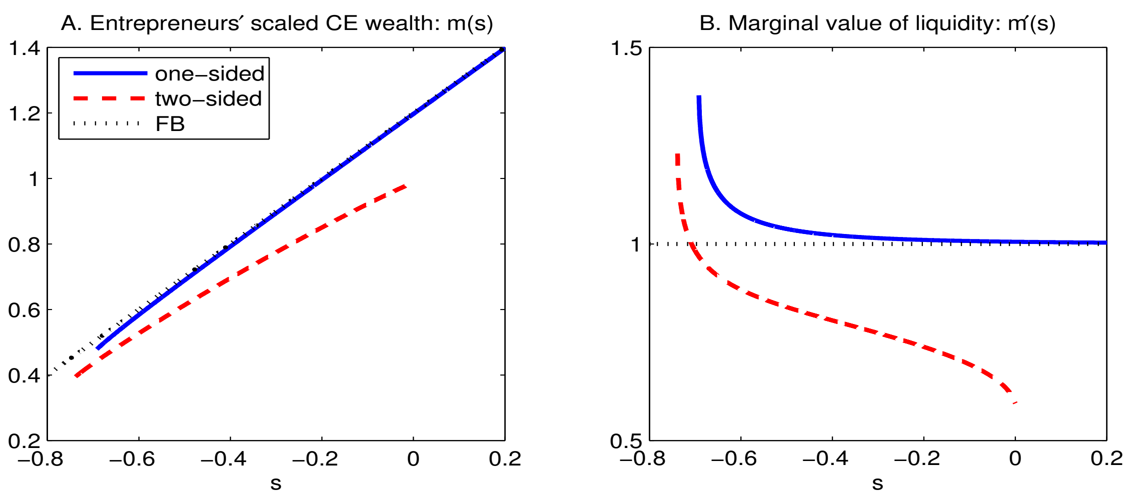
For the one-sided limited-commitment problem (by the entrepreneur), as $w \to \infty$, $p(w)$ approaches $q^{FB} - w$ and $p'(w) \to -1$, the first-best MM benchmark result as in Hayashi (1982). Importantly, the entrepreneur's inability to fully commit not to walk away *ex post* puts a lower bound $\underline{w}$ for w. In our numerical example, $w \geq \underline{w} = 0.479$.

For the two-sided limited-commitment case, w lies between $\underline{w} = 0.40$ and $\overline{w} = 0.988$. The upper boundary $\overline{w}$ is determined by $p(\overline{w}) = 0$ as the entrepreneur will face zero liability after following the deviation strategy by diverting α fraction of capital stock. Interestingly, the left boundary for the two-sided case, $\underline{w} = 0.40$, is lower than that for the one-sided case, $\underline{w} = 0.479$. Intuitively, the additional constraint due to the investors' limited liability not only restricts the support of w to be lower than $\overline{w} = 0.988$, but also shifts the left boundary $\underline{w}$ further to the left from 0.479 for the one-sided case to 0.40, as the entrepreneur's outside opportunity (by following the deviation strategy) is now also less attractive.

While $p'(w) \geq -1$ holds for the one-sided case, $p'(w)$ can be less than -1 for the two-sided case due to the fact the benefit for an entrepreneur from an increase of w may not be sufficient to offset the cost to the investor (due to the increased likelihood that the investors' limited liability constraint may bind in the future) implying $p'(w) + 1 < 0$.

It is worth noting that despite being risk neutral, the investor effectively behaves in a risk-averse manner due to the entrepreneur's limited enforcement and/or the investors' limited liability, as we see from the concavity of the investors' scaled value $p(w)$. This concavity property is critical in our agency model and fundamentally differentiates our model from the neoclassical Hayashi (1982) result where volatility has no effect on firm value.

Figure 2: **The entrepreneur's scaled certainty equivalent wealth $m(s)$ and marginal (certainty equivalent) wealth of s, $m'(s)$, as functions of scaled liquidity s.** The solid and dashed lines correspond to the one-sided and two-sided limited-commitment cases, respectively. For both cases, $m(s)$ is increasing and concave. For the one-sided case, $s \geq= -\overline{d} = \underline{s} = -0.692$. For the two-sided case, $-0.738 = -\overline{d} = \underline{s} \leq s \leq 0$. The dotted line depicts the first-best MM results: $m(s) = q^{FB} + s$ and the sensitivity $m'(s) = 1$.



Next, we provide an intuitive implementation of our contracting problem by expressing the entrepreneur's scaled promised certainty equivalent wealth w as $m(s)$, a function in the entrepreneur's liquidity s, in that $w = m(s)$.

Scaled liquidity s and the entrepreneur's scaled certainty-equivalent wealth $m(s)$. Figure 2 plots $m(s)$ and the marginal value $m'(s)$ in Panels A and B, respectively. Note that $m(s)$ is increasing in s for both the one-sided and two-sided limited-commitment cases. Intuitively, the higher the value of liquidity s, the less likely that the entrepreneur will walk away and hence the higher the value of $m(s)$ in the long-term bilateral relationship.

Moreover, as s increases, the entrepreneur becomes less constrained and the marginal value $m'(s)$ decreases.

For the one-sided limited-commitment case (by the entrepreneur), as $s \to \infty$, the entrepreneur's scaled (certainty equivalent) wealth approaches $q^{FB} + s$ and $m'(s) \to 1$, the first-best MM benchmark result as in Hayashi (1982).[16] Importantly, the entrepreneur's inability to fully commit not to walk away *ex post* puts a lower bound $\underline{s}$ for s. How much risk-free debt can the entrepreneur borrow without defaulting on the liability? We refer to the maximal amount of risk-free debt that the entrepreneur can borrow also as the endogenous risk-free debt capacity, denoted by $\overline{d} = \overline{D}(K)/K$. In our numerical example, debt is risk-free provided that $s > -\overline{d} = -0.692$. Intuitively, the entrepreneur can owe the investors up to $p(\underline{w})$, the investors' value when the entrepreneur's participation constraint binds. Hence, the maximal amount of debt, i.e. the debt capacity satisfies $\overline{d} = p(\underline{w})$.

For the two-sided limited-commitment case, s lies between $\underline{s} = -0.738$ and $\overline{s} = 0$, which implies that the entrepreneur will be able to borrow up to $\overline{d} = 0.738$ but will not have any savings in this implementation. The upper boundary is $\overline{s} = 0$. If $\overline{s} > 0$, the investors' value is strictly negative violating the investors' limited-liability condition. Interestingly, the lower boundary for the two-sided case, $\underline{s} = -\overline{d} = -0.738$, is to the left of the lower boundary for the one-sided case, $\underline{s} = -\overline{d} = -0.692$. Intuitively, the additional constraint due to investors' limited-liability condition limits the entrepreneur's self savings capacity, which in turn increases the entrepreneur's demand of using credit line, an alternative liquidity instrument causing an increase of the firm's debt capacity $\overline{d}$ from 0.692 to 0.738. This is another example of understanding the rich implications of endogeneity. Here, a firm with a larger debt capacity is not necessarily less constrained and may have a lower value.

While $m'(s) \geq 1$ holds for the one-sided case, $m'(s)$ can be less than 1 for the two-sided case. This is again due to the fact that in the two-sided case the benefit of relaxing financial constraints for the entrepreneur from an increase of s may not be sufficient to offset the cost to the investor (due to a shorter distance investors' limited-liability constraint) implying $m'(s) < 1$ in the region $-0.708 < s \leq 0$. We next analyze the optimal policy rules.

[16] See Wang, Wang, and Yang (2012) for similar conditions in a model with exogenously-specified incomplete-markets model of entrepreneurship.

6.4 Investment, Consumption, Liquidity and Risk Management

We first analyze the firm's investment decisions, then the entrepreneur's consumption, and finally corporate liquidity and risk management decisions.

6.4.1 Investment, marginal q, and marginal value of liquidity $m'(s)$.

We may simplify the FOC for investment as follows,

$$1 + g'(\mathfrak{i}(s)) = \frac{J_K}{J_S} = \frac{M_K}{M_S} = \frac{m(s) - sm'(s)}{m'(s)} \,. \tag{49}$$

where the first equality is the investment FOC, the second equality follows from the value function given by (36), and the last equality follows from the homogeneity property of M in K and S. First recall the first-best MM benchmark result. Under perfect capital markets, the entrepreneur's certainty equivalent wealth $M(K,S) = m(s)K = (q^{FB} + s)K$ and the marginal value of liquidity $M_S = 1$ at all times. Hence, (49) specializes to the Hayashi's condition, where the marginal cost of investing $1 + \theta\mathfrak{i}(s)$ equals the marginal q.

With limited commitment frictions, $M_S \neq 1$ in general and the FOC (49) then states that the marginal cost of investing (on the left side) equals the ratio between (a) the marginal q, measured by M_K, and (b) the marginal value of liquidity measured by M_S. While financing does not matter in the standard q theory of investment under perfect capital markets, here financing is costly and we use M_S to measure the (endogenous) marginal cost of financing in our generalized q theory of investment where markets are endogenously incomplete.

Figure 3 demonstrates the effects of limited commitments on marginal q and optimal investment $\mathfrak{i}(s)$. The dotted lines in Panels A and B of Figure 3 give the first-best $q^{FB} = 1.2$ and $i^{FB} = 0.1$, respectively. For the one-sided limited-commitment case, the investment-capital ratio $\mathfrak{i}(s)$ is lower the first-best benchmark value, $i^{FB} = 0.1$ for all s and increases from 0.02 to $i^{FB} = 0.1$, as we increase s from the left boundary $\underline{s} = -0.692$ towards ∞. This is a standard result that increasing liquidity mitigates the severity of under-investment (compared with the first-best benchmark level) by a financially constrained firm.

However, surprisingly, marginal q, M_K, *decreases* in s from 1.43 to 1.197 in the credit region $s < 0$. What is the intuition? When the firm is financing its investment via credit on the margin (i.e., $S < 0$), increasing K moves a negative-valued s closer to the origin thus mitigating financial constraints, which is an additional benefit of accumulating K. Technically, this result follows from $dM_K/ds = -sm''(s) < 0$ when $s < 0$ and if $m(s)$ is concave.

But why does a high marginal-q firm invest less in the credit region $s < 0$? And how do we reconcile an increasing investment function $\mathfrak{i}(s)$ with a decreasing marginal q function, $M_K = m(s) - sm'(s)$ in the credit region $s < 0$? It is because in the credit region $s < 0$, a high marginal-q firm also faces a high financing cost. In our case, marginal q and the marginal financing cost are perfectly correlated. And investment is determined by the ratio between marginal q and marginal financing cost as we have noted. For example, at the left boundary $\underline{s} = -\overline{d} = -0.692$, marginal q is 1.43 and the marginal value of liquidity, $m'(s)$, is 1.38 both of which are high. However, together they imply $\mathfrak{i}(-0.692) = 0.02$, which is a very low compared with the first-best $i^{FB} = 0.10$.

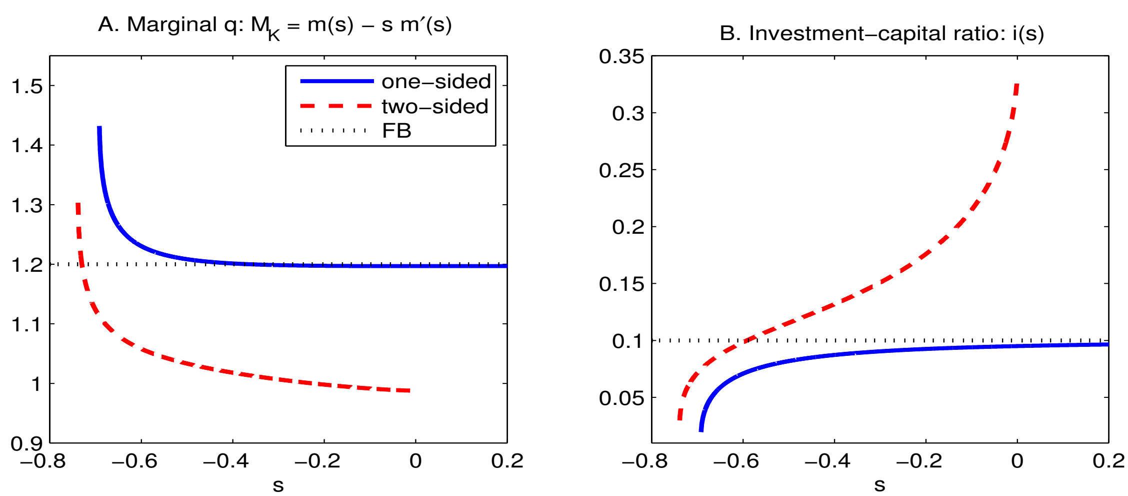


Figure 3: **Marginal q, $M_K = m(s) - sm'(s)$, and the investment-capital ratio $\mathfrak{i}(s)$.** The solid and dashed lines correspond to the one-sided and two-sided limited-commitment cases, respectively. For the one-sided case, the firm always under-invests and $\mathfrak{i}(s)$ increases with s. For the two-sided case, the firm may either under-invest or over-invest. For $-0.591 < s \leq 0$, the firm over invests due to the investors' limited-liability condition. The dotted line depicts the first-best MM results where $M_K = q^{FB} = 1.2$ and the investment-capital ratio $\mathfrak{i}(s) = i^{FB} = 0.1$.

For space considerations, we do not elaborate on the properties of marginal q, M_K, and marginal value of liquidity, and investment in the cash region $s > 0$. In that region, M_K and

$m'(s)$ are negatively correlated, different from the positive correlation in the credit region.[17]

Next, we turn to the two-sided case. First, the entrepreneur cannot own the whole equity of the productive asset and also has positive liquid wealth; Otherwise, the investors' value would be strictly negative violating the limited-lability condition. Hence, there is only credit region for the two-sided case: $s \leq 0$.

Second, the firm may either under-invest or over-invest compared with the first-best benchmark. In our numerical example, the firm under-invests when $s < -0.591$ but over-invests when $-0.591 < s \leq 0$. Intuitively, whether the first under-invests or over-invests depends on the net effects from the entrepreneur's limited-commitment and the investors' limited-liability constraints. For sufficiently low values of s (e.g., deep in debt), the entrepreneur's participation constraint matters more and hence the firm under-invests as it does in the one-sided case. However, for sufficiently high values of s (bearing little debt and being close to the origin), the investors' value is close to zero and hence the investors' limited-liability constraint has a stronger influence on investment. In order to make sure that s will drift back into the credit region, the entrepreneur needs to "save" in the illiquid productive asset by increasing K and borrowing more. By over-investing, the firm optimally manages to keep s between $-\overline{d}$ and 0. The fact that we observe firms' excessive investment near bankruptcy (when investors' equity is barely positive) may not indicate that the firm is engaging in excessive risk taking but simply to meet the investors' limited-liability constraint.

Having analyzed the firm's investment, we next turn to the entrepreneur's consumption.

6.4.2 Consumption

The entrepreneur's optimal consumption rule $\mathfrak{c}(s)$ is given by

$$\mathfrak{c}(s) \quad = \quad \chi m'(s)^{-1/\gamma} m(s) \, , \tag{50}$$

where χ is the MPC (as in Ramsey) given by (16). Figure 4 plots the optimal consumption-capital ratio $\mathfrak{c}(s)$, and the MPC $\mathfrak{c}'(s)$ in Panels A and B, respectively.

[17] See Bolton, Chen, and Wang (2011) for discussions on how cash and credit influence the behaviors of investment, marginal q, and marginal value of liquidity.

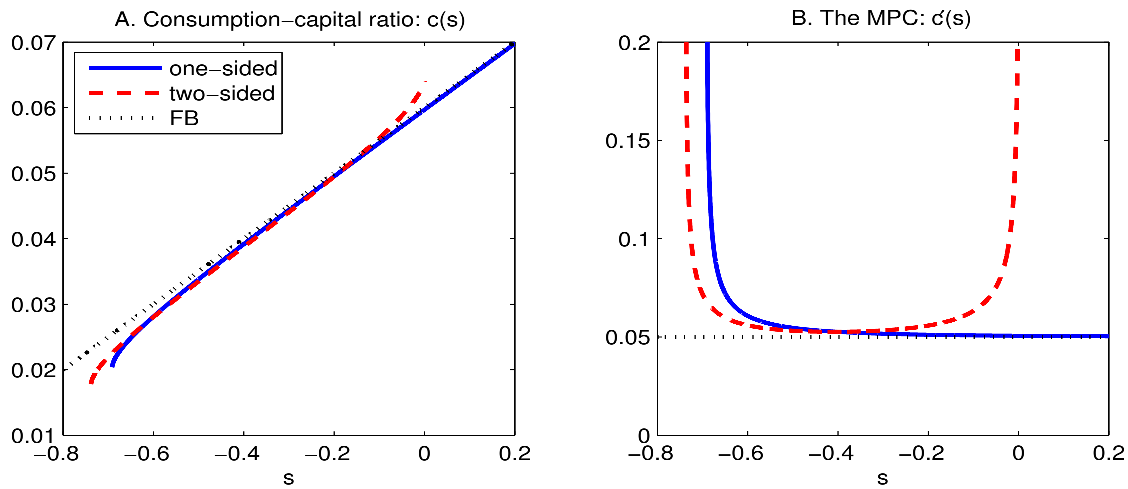


Figure 4: **Consumption-capital ratio $\mathfrak{c}(s)$ and the MPC $\mathfrak{c}'(s)$.** The solid and dashed lines correspond to the one-sided and two-sided limited-commitment cases, respectively. For the one-sided case, the entrepreneur always under-consumes and $\mathfrak{c}(s)$ increases with s. For the two-sided case, the entrepreneur may either under-consume or over-consume. In our two-sided example, for $-0.131 < s \leq 0$, the entrepreneur over-consumes due to the investors' limited-liability condition. The dotted line depicts the first-best permanent-income results: $\mathfrak{c}(s) = \chi(s + q^{FB})$ and MPC $\mathfrak{c}'(s) = \chi = 5\%$.

For both one-sided and two-sided limited-commitment cases, the higher the value of liquidity s, the higher the entrepreneur's consumption, i.e., $\mathfrak{c}(s)$ is increasing in s, as seen in the figure.

For the one-sided limited-commitment case, as $s \to \infty$, $m(s) \to q^{FB} + s$, the marginal value of liquidity $m'(s) \to 1$, and therefore $\mathfrak{c}(s) \to \chi\left(q^{FB} + s\right)$, the permanent-income benchmark result. For finite values of liquidity, i.e., $-\overline{d} \leq s < \infty$, consumption $\mathfrak{c}(s)$ is strictly below the first-best permanent-income benchmark (see the dotted line), as we see from Panel A. The MPC $\mathfrak{c}'(s)$ decreases significantly with s and approaches the permanent-income benchmark $\chi = 5\%$ as we increase $s \to \infty$. That is, financially constrained entrepreneurs deep in debt (i.e., s being close to $-\overline{d}$) have MPCs that are substantially higher than the permanent-come benchmark. For the one-sided case, the consumption function induced by the limited enforcement friction is concave, which is consistent with the concave consumption function in standard exogenously-specified incomplete-markets environments.

For the two-sided limited-commitment case, the entrepreneur's consumption can be either below or above the first-best benchmark consumption rule. In our numerical example, compared with the permanent-income benchmark, the entrepreneur under-consumes for $s < -0.131$ but over-consumes for $-0.131 < s \leq 0$. At $s = 0$, $\mathfrak{c}(0) = 6.41\%$, which is greater than $\mathfrak{c}^{FB}(0) = \chi q^{FB} = 6\%$.

Intuitively, whether the entrepreneur under-consumes or over-consumes depends on the net effects from the entrepreneur's limited-commitment and the investors' limited-liability constraints. Intuitively, for sufficiently low values of s (e.g., deep in debt), the entrepreneur's participation constraint matters more and hence the entrepreneur under-consumes in order to build up s as the entrepreneur does in the one-sided limited-commitment case. However, for sufficiently high values of s (bearing little debt and being close to the origin) the investors' value is close to be zero and hence the investors' limited-liability constraint has a stronger influence on the entrepreneur's consumption. In order to make sure that s will drift back into the credit region, the entrepreneur needs a high consumption rate to lower the rate of paying down the credit line. By over-consuming for sufficiently high s, the entrepreneur optimally manages to keep s between $-\overline{d}$ and 0. Also note that the MPC $\mathfrak{c}'(s)$ is not monotonic in liquidity s due to the interactions between the entrepreneur's limited commitment and the investors' limited-liability condition.

Next we turn to the firm's optimal financial policies.

6.4.3 Hedging via Futures

Before we delve into the details of corporate liquidity and risk management, we first review the entrepreneur's total wealth portfolio which consists of three parts: (1) a 100% equity in the underlying business; (2) a zero-value mark-to-market futures position; and (3) a liquidity asset holding in the amount of s (possibly negative in which case means borrowing.)

The entrepreneur's optimal futures position $\boldsymbol{\phi}(s)$ is given by

$$\boldsymbol{\phi}(s) \quad = \quad \frac{sm''(s)m(s) + \gamma m'(s)(m(s) - sm'(s))}{m(s)m''(s) - \gamma m'(s)^2} \,. \tag{51}$$

By construction, the only liquid risky asset in this implementation is futures, which is marked to market and has zero market value at all times, therefore, all corporate liquidity s must be held in the risk-free asset. (When $s < 0$, liquidity refers to the credit borrowed by the entrepreneur from the investors.)

Figure 5 plots the futures position $\phi(s)$. First, we note that under the first-best MM benchmark, the entrepreneur is fully insured from the capital shock by taking a short futures position with a short position, $\phi(s) = -q^{FB} = -1.2$. See the dotted line for the first-best complete hedging results for the entrepreneur with $\phi(s) = -q^{FB} = -1.2$ in Figure 5.

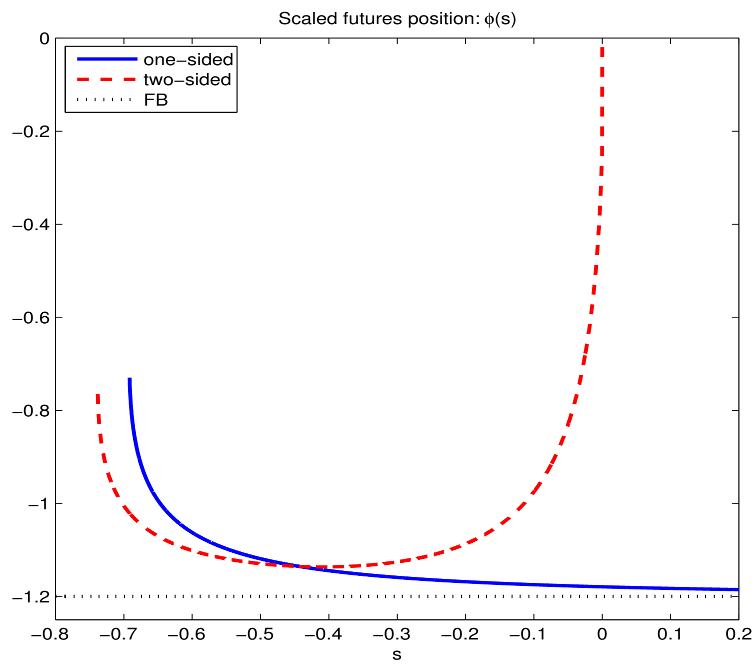


Figure 5: **Futures hedging position $\phi(s)$ and savings s.** The solid and dashed lines correspond to the one-sided and two-sided limited-commitment cases, respectively. For both cases, the entrepreneur takes a short position in the futures to partially hedge the equity exposure to the underlying business, in that $\phi(s) < 0$. For the one-sided case, interestingly, the total exposure $|\phi(s)|$ increases with s. For the two-sided case, $|\phi(s)|$ is non-monotonic in s due to the interaction between the entrepreneur's limited-commitment constraint and the investors' limited-liability condition. The dotted line depicts the first-best complete hedging results for the entrepreneur with $\phi(s) = -q^{FB} = -1.2$.

For both limited-commitment cases, the entrepreneur takes a short position in the futures to partially hedge the equity exposure to the underlying business, in that $\phi(s) < 0$. How does $|\phi(s)|$ depend on s? For the one-sided limited-commitment case, as the firm becomes less constrained (i.e. as s increases), the entrepreneur increases the (absolute) size of the futures hedging position measured by $|\phi(s)|$. That is, less financially constrained firms hedge more (after controlling for firm size) and in the limit as $s \to \infty$, the entrepreneur can fully diversity the idiosyncratic business risk by taking a short futures position with a size of $-q^{FB} = -1.2$ achieving the first-best MM benchmark. Rampini, Sufi, and Viswanathan (2013) document less constrained firms hedge more.

For the two-sided limited-commitment case, the entrepreneur's futures hedging position $|\boldsymbol{\phi}(s)|$ is non-monotonic in s where s lies between $\underline{s} = -0.738$ and $\overline{s} = 0$. By requiring $\sigma^s(s) = 0$ at both boundaries $\underline{s} = -0.738$ and 0, we have $\boldsymbol{\phi}(\underline{s}) = \underline{s} = -0.738$ and $\boldsymbol{\phi}(0) = 0$ by using (40). Therefore, the firm optimally chooses not to hedge at $s = 0$ as it fully pays back all its credit. Note that the maximal amount of hedge is $|\boldsymbol{\phi}| = 1.137$ at $s = -0.408$. Intuitively, in the two-sided limited-commitment case, the maximal hedge occurs at an interior value of s as the entrepreneur minimizes the total costs of financial frictions due to the entrepreneur's limited-commitment and the investors' limited-liability frictions.

One general take-away message from this comparative analysis between the one-sided and two-sided limited-commitment cases is that the investors' limited-liability constraint can have important implications that are very different from and sometimes opposite to those implied by the entrepreneur's limited-commitment constraint.

6.5 Risk Management via Shorting Stocks: An Alternative

Next, we provide an alternative implementation achieving the same resource allocation as the previous implementation (based on the bank savings/credit account and futures) does. However, this implementation has very different implications on debt capacity.

For the new implementation, we introduce a risky liquid financial asset that is *perfectly correlated* with the shock Z for capital accumulation (1) so that the entrepreneur can choose how much capital shock Z to hedge. Let dR_t denote the incremental return for this risky asset over time period $(t, t+dt)$. Because investors are risk neutral, by using the standard equilibrium argument, we may write down dR_t as follows,

$$dR_t = rdt + \sigma_K dZ_t \,. \tag{52}$$

Without loss of generality, we choose the volatility of this new risky asset to be σ_K. By setting the volatility of this risky asset to equal to the volatility of capital accumulation process, we essentially are requiring a unit short position in the risky asset provides an instantaneous perfect hedging against capital accumulation risk.

Let Ω_t denote the entrepreneur's investment in this new risky asset, and hence the remaining liquid wealth, $S_t - \Omega_t$, is invested in the entrepreneur's savings account earning the risk-free asset r. Because the entrepreneur can costlessly and continuously rebalance between this new publicly traded risky asset and the risk-free asset, we may write the evolution for the entrepreneur's total liquid wealth S_t as follows,

$$\begin{aligned} dS_t &= (r(S_t - \Omega_t) + Y_t - C_t)dt + \Omega_t(rdt + \sigma_K dZ_t) \\ &= (rS_t + Y_t - C_t)dt + \Omega_t \sigma_K dZ_t \,. \end{aligned} \tag{53}$$

By comparing (53) with (34) (and ignoring the jump part in (34)), it is straightforward to conclude that the stock's position $\omega(s) = \Omega/K$ in this new implementation is the same as the futures position $\boldsymbol{\phi}(s)$ in the previous implementation, i.e. $\omega(s) = \boldsymbol{\phi}(s)$. Unlike futures, the entrepreneur collects the short-sale proceeds, $-\Omega$, and invests the net proceeds $S - \Omega$ (after paying down the credit usage amount) in the savings account earning interests at the risk-free rate r.

Figure 6 plots the hedging position via the risky liquid asset $\omega(s)$ and the amount of risk-free asset holdings, $s - \omega(s)$, which earns interests at the risk-free rate r in Panels A and B, respectively. First, as we have noted, the risky asset position is the same as the futures hedging position, i.e. $\omega(s) = \boldsymbol{\phi}(s)$ because the risky asset and futures (on the risky asset) have the same risk exposures $\sigma_K dZ$. As for the futures, the entrepreneur needs to take a short position in the risky asset whose return is given by (52) in order to partially manage the risk exposure to the underlying illiquid business project.

While the savings amount under the futures hedging is simply s, the scaled savings amount in this new implementation equals $s - \omega(s) \neq s$ as shorting $\omega(s)$ shares of stocks (per unit of capital) generate a sales proceed in the amount of $-\omega(s) > 0$. Panel B of Figure 6 shows that for both cases, the entrepreneur stochastically saves, i.e., $s - \omega(s) > 0$. For the one-sided case, scaled savings $s - \omega(s)$ increases from zero to $s + q^{FB}$ in the limit as we increase liquidity s from -0.692 towards ∞. For the two-sided case, risk-free savings

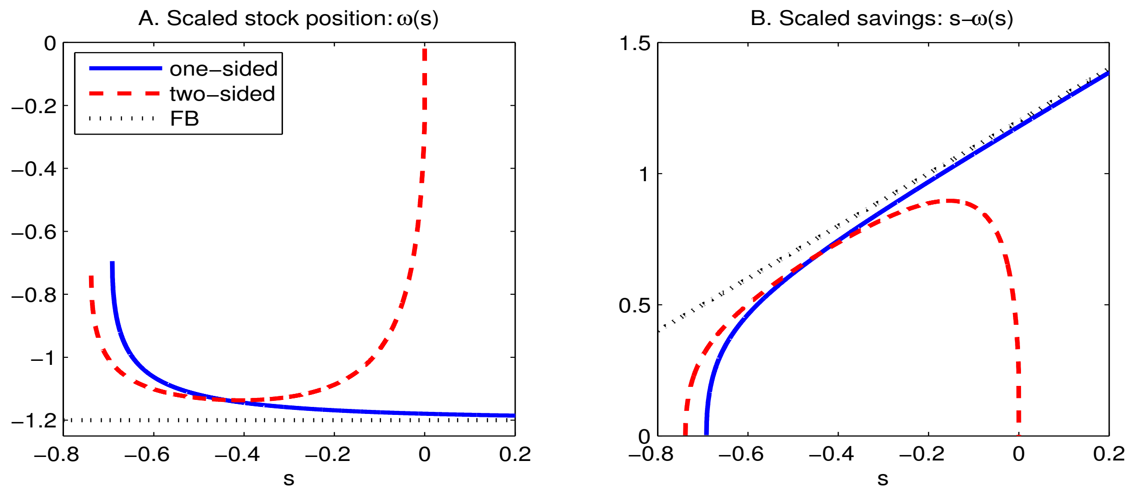


Figure 6: **Optimal hedge via the risky asset $\omega(s)$ and savings $s - \omega(s)$.** The solid and dashed lines correspond to the one-sided and two-sided limited-commitment cases, respectively. For both cases, the entrepreneur takes a short position in the risky liquid asset to partially hedge the equity exposure to the underlying business, in that $\omega(s) < 0$, and holds a risk-free savings account with (weakly) positive balances at all times. For the one-sided case, interestingly, the total size of the "short" position $|\omega(s)|$ increases with s. For the two-sided case, $|\omega(s)|$ is non-monotonic in s due to the interaction between the entrepreneur's limited-commitment constraint and the investors' limited-liability condition. The dotted line depicts the first-best MM results: $\omega(s) = -q^{FB} = -1.2$ and savings $s + q^{FB}$.

$s - \omega(s)$ is non-monotonic. At both left and right boundaries $s = \underline{s} = -0.738$ and $s = 0$, liquid savings $s - \omega(s) = 0$ equal zero, which follows from the requirement that volatility $\sigma^s(s) = -(s - \omega(s))\sigma_K$ at the boundaries must be zero. In the interior region $-0.738 < s < 0$, the savings amount, $s - \omega(s)$, first increases and then decreases with s essentially inversely tracking the non-monotonicity of the hedge ratio $\omega(s)$.

While using different securities, the two implementations share two key features in common: (1) the total corporate liquidity summarized by s and (2) the total amount of risk exposures.

7 Alternative Specifications of Outside Options for Investors and Entrepreneur

The critical assumptions of our framework are (i) the inalienability of human capital and (ii) the investors' inability to fund the operating losses indefinitely. For expositional simplicity, the specific constraints we have chosen are (i) the entrepreneur's ability to divert α fraction of capital and start a new firm and (ii) the limited-liability constraint for investors at all times. However, it is important to note that our model's main results and key insights hold under much broader settings. We next consider (i) an alternative specification that will pin down the entrepreneur's outside option and (ii) one important generalization on investors' limited-liability constraint along the line of Hart and Moore (1994).

7.1 Autarky as the Entrepreneur's Outside Option

Model setup. Now we provide an alternative interpretation for the entrepreneur's outside option based on the cost of losing intertemporal consumption-smoothing opportunities. Instead of assuming that the entrepreneur can divert α fraction of capital stock and start afresh, we assume that the entrepreneur always has an option to freely walk away from the investors. However, by doing so, the entrepreneur will permanently lose all future borrowing, saving, and insurance possibilities by remaining in autarky, as assumed in Bulow and Rogoff (1989) and the follow-up international macro literature.

Let $\widehat{J}(K_t)$ denote the entrepreneur's value function under autarky defined as follows,

$$\widehat{J}(K_t) = \max_{I} \ \mathbb{E}_t \left[\int_t^{\infty} \zeta e^{-\zeta(v-t)} U(Y_v) dv \right] , \tag{54}$$

where the entrepreneur's consumption is given by $C_t = Y_t = A_t K_t - I_t - G_t$. The following proposition summarizes the entrepreneur's value function $\widehat{J}(K)$ and the certainty equivalent wealth $\widehat{M}(K)$ under autarky for the pure diffusion case.[18]

[18] See the Appendix for technical details.

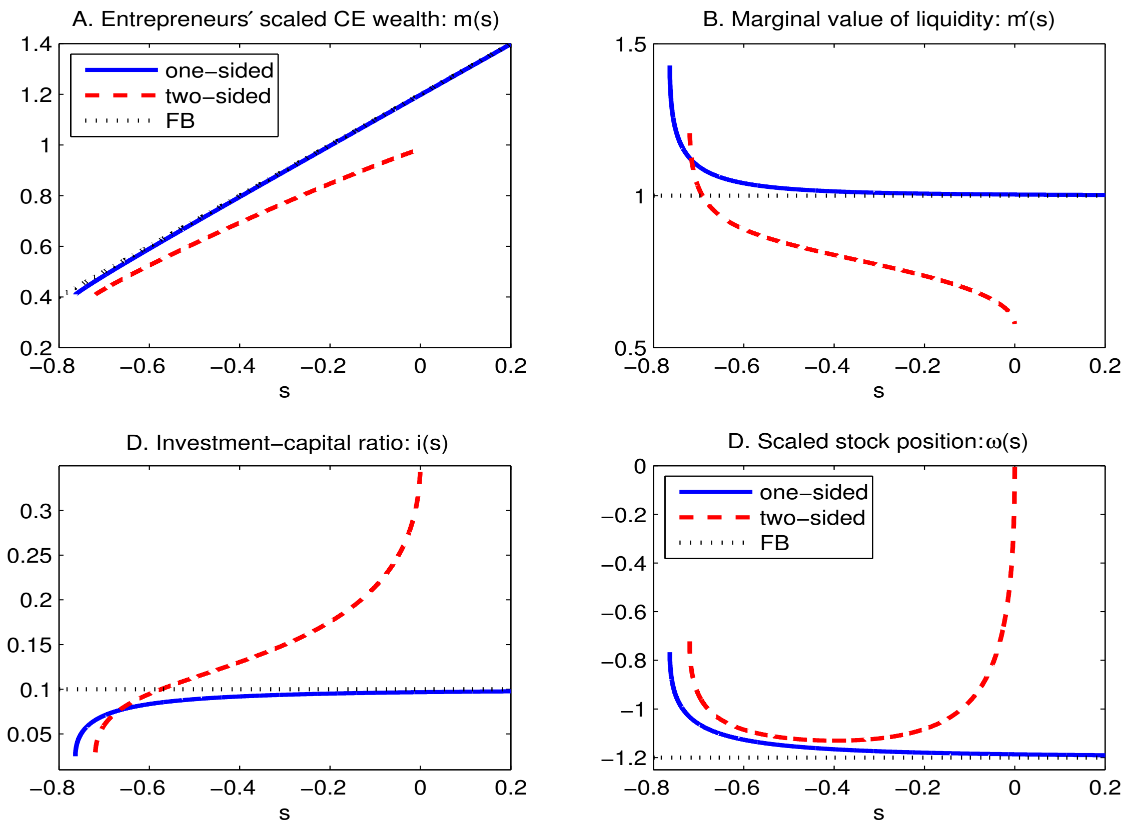


Figure 7: **The case with autarky as the entrepreneur's outside option.** Panels A, B, C, and D plot the entrepreneur's scaled certainty equivalent wealth $m(s)$, marginal (certainty equivalent) wealth of s, $m'(s)$, the investment-capital ratio $\mathfrak{i}(s)$, and optimal hedge via the risky asset $\omega(s)$, as functions of scaled liquidity s, respectively. The solid and dashed lines correspond to the one-sided and two-sided limited-commitment cases, respectively. For both cases, $m(s)$ is increasing and concave. For the one-sided case, $s \geq -0.764$. For the two-sided case, $-0.720 \leq s \leq 0$. The dotted line depicts the first-best MM results: $m(s) = q^{FB} + s$, the sensitivity $m'(s) = 1$, the investment-capital ratio $\mathfrak{i}(s) = i^{FB} = 0.1$ and $\omega(s) = -q^{FB} = -1.2$.

Proposition 3 *Under autarky, the entrepreneur's value function $\widehat{J}(K)$ is given by*

$$\widehat{J}(K) = \frac{(b\widehat{M}(K))^{1-\gamma}}{1-\gamma}\,, \tag{55}$$

where b is given by (14) and $\widehat{M}(K)$ is the entrepreneur's certainty equivalent wealth given by

$$\widehat{M}(K) = \widehat{m}K\,, \tag{56}$$

where

$$\widehat{m} = \frac{(\zeta(1+g'(\widehat{i}))(A-\widehat{i}-g(\widehat{i}))^{-\gamma})^{\frac{1}{1-\gamma}}}{b}, \tag{57}$$

and $\widehat{i}$ *is the optimal investment-capital ratio solving the following implicit equation:*

$$\zeta = \frac{A-\widehat{i}-g(\widehat{i})}{1+g'(\widehat{i})} + (\widehat{i}-\delta)(1-\gamma) - \frac{\sigma_K^2\gamma(1-\gamma)}{2}. \tag{58}$$

By following essentially the same analysis in Section 6, we conclude that the lower boundary $\underline{s}$ for liquidity s is determined by:

$$m(\underline{s}) = \widehat{m}. \tag{59}$$

Therefore, for both the one-sided and two-sided limited-commitment cases, we only need to replace the previous boundary condition (43) in Proposition 2 with the new condition boundary condition (59) and keep all the other conditions are kept unchanged. Intuitively, the lower boundary $\underline{s}$ for this new case is determined solely by (59) independent of the upper boundary $\underline{s}$, which is very different from the benchmark specification where the entrepreneur can divert α fraction of capital stock and start a new firm free of liability.

Analysis. Figure 7 plots the entrepreneur's scaled certainty equivalent wealth $m(s)$, the marginal value of liquidity $m'(s)$, optimal investment-capital ratio $\mathfrak{i}(s)$, and optimal hedging position (scaled stock position) $\omega(s)$ for both one-sided and two-sided limited-commitment cases. The general patterns for all four variables remain valid. For example, for the one-sided case, the firm always under-invests and the marginal value of liquidity $m'(s)$ is always greater than one. Additionally, the degree of underinvestment weakens and the marginal value of liquidity $m'(s)$ decreases, both of which eventually approach the first-best levels $i^{FB} = 0.10$ and unity, respectively, as $s \to \infty$. Finally, the optimal hedge size, $|\phi(s)|$ also increases and approaches the first-best level q^{FB} for $s \to \infty$.

7.2 Investors' Alternative Use of Capital (Hart and Moore, 1994)

We now consider a more general specification for the investors' outside option. Suppose that investors' alternative use of capital can yield a value of ℓK_t at any time t, where $\ell > 0$ is a constant. For example, investors may simply liquidate the asset in the market or hire a less skilled manager delivering ℓK to investors. As a result, *ex ante* investors cannot credibly commit to a long-term contract when investors' value may fall below ℓ per unit of capital on the equilibrium path effectively making the investors' participation constraint even tighter. Indeed, this latter case is the assumption on the investors' side in Hart and Moore (1994).

Additionally, we could imagine that investors may be able to provide some personal collateral *ex ante* for the entrepreneur in an escrow account that can be seized by the entrepreneur should investors walk away *ex post* from the long-term contract. By personally pledging more capital in advance mitigates the investors' incentives to walk away from the optimal contract.

In the model, we simply need to modify the upper boundary condition as follows:

$$F(K, \overline{V}(K)) \geq \ell K \,, \tag{60}$$

where $\overline{K}$ is the endogenous upper boundary. The case with $\ell > 0$ corresponds to the Hart-Moore framework where the investors have an alternative use of capital. For the case with $\ell < 0$, we may interpret ℓ as the amount of personal guarantee offered by investors *ex ante* in an escrow account that can be seized by the entrepreneur should the investors renege on the contract.

8 Persistent Productivity Shocks: Insurance and Defaultable Debt

In this section, we consider the model's general case by allowing for persistent observable shocks to the firm's productivity. First, it is natural to assume that these productivity shocks are observable and can be contracted on. Second, because these productivity shocks are persistent naturally they will affect the firm's investment even in the neoclassical setting as expected. We will show there is an additional interaction effect due to financing constraint and persistent productivity shocks.[19]

We explore the interaction effect of persistent productivity shocks and the entrepreneur's limited commitment and the consequences for investment, consumption, managerial compensation, and liquidity and risk management. As we will show, persistent productivity shocks will naturally give rise to demand for insurance against the change of productivity. Equivalently, we show that default on debt as productivity decreases from H to L can be a natural equilibrium outcome.

We leave the solution for the optimal contracting problem to the Appendix and focus on an intuitive financial implementation with commonly used securities.

8.1 Implementation: Liquidity and risk management

First, by using the homogeneity property, we write the entrepreneur's certainty equivalent wealth function in state $n \in \{L, H\}$, $M(K, S, A^n)$, as follows,

$$M(K, S, A^n) = m_n(s)K\,. \tag{61}$$

The dynamic of scaled liquidity s. Given the state-contingent consumption-capital ratio $\mathfrak{c}_n(s)$, the investment-capital ratio $\mathfrak{i}_n(s)$, the hedge ratio $\boldsymbol{\phi}_n(s)$ and the endogenous adjustment size $\pi_n(s)$ of liquidity holding as productivity switches out of state n, we write the dynamic of liquidity s in the interior region as follows,

$$ds_t = \mu_n^s(s_t)dt + \sigma_n^s(s_t)dZ_t + \pi_n(s)dN_t\,, \tag{62}$$

where the drift and volatility processes $\mu^s(\,\cdot\,)$ and $\sigma^s(\,\cdot\,)$ for s are given by

$$\begin{aligned}
\mu_n^s(s) &= (A^n - \pi_n(s)\lambda_n - \mathfrak{i}_n(s) - g(\mathfrak{i}_n(s)) - \mathfrak{c}_n(s)) + (r + \delta - \mathfrak{i}_n(s))s - \sigma_K\sigma_n^s(s), && (63)\\
\sigma_n^s(s) &= (\boldsymbol{\phi}_n(s) - s)\sigma_K\,. && (64)
\end{aligned}$$

[19]See DeMarzo, Fishman, He, and Wang (2012) for a model of optimal investment in a q-theoretic context with persistent shocks and agency frictions along the line of DeMarzo and Fishman (2007) and DeMarzo and Sannikov (2006).

Here, the last term in (62), $\pi_n(s)dN_t$, captures the effect of discrete productivity change on scaled liquidity s. $\pi_n(s)$ is the scaled insurance position.

The one-sided limited-commitment case. The following proposition summarizes the solution for the case with only the entrepreneur's limited-commitment problem.

Proposition 4 *In the region $s > \underline{s_H}$, the scaled value $m_H(s)$ in state H solves the following ODEs,*

$$\begin{aligned} 0 &= \max_{\mathrm{i}_H;\pi_H} \frac{m_H(s)}{1-\gamma}\left[\gamma\chi m_H'(s)^{\frac{\gamma-1}{\gamma}} - \zeta\right] - \delta m_H(s) + \left[(r+\delta)s + A^H - \lambda_H\pi_H\right] m_H'(s) \\ &\quad + \mathrm{i}_H(m_H(s) - (s+1)m_H'(s)) - g(\mathrm{i}_H)m_H'(s) - \frac{\gamma\sigma_K^2}{2}\frac{m_H(s)^2 m_H''(s)}{m_H(s)m_H''(s) - \gamma m_H'(s)^2} \\ &\quad + \frac{\lambda_H m_H(s)}{1-\gamma}\left(\left(\frac{m_L(s+\pi_H)}{m_H(s)}\right)^{1-\gamma} - 1\right), \end{aligned} \tag{65}$$

subject to the following boundary conditions:

$$\lim_{s\to\infty} m_H(s) = q_H^{FB} + s\,, \tag{66}$$

$$m_H(\underline{s_H}) = \alpha m_H(0)\,, \tag{67}$$

$$\lim_{s\to\underline{s_H}} \sigma_H^s(s) = 0 \quad \text{and} \quad \lim_{s\to\underline{s_H}} \mu_H^s(s) \geq 0\,. \tag{68}$$

The underlying arguments are very similar to the ones for the pure-diffusion case.

The two-sided limited-commitment case. For this two-sided limited commitment problem, we simply need to modify the condition at the the upper boundary in Proposition 4. Note that the upper boundary is $s = 0$ rather than the natural limiting boundary $s \to \infty$ for the one-sided limited-commitment case. We thus replace condition (66) with the following conditions at the new upper boundary $s = 0$ under state H:

$$\lim_{s\to 0} \sigma_H^s(s) = 0 \quad \text{and} \quad \lim_{s\to 0} \mu_H^s(s) \leq 0\,. \tag{69}$$

The arguments for (69) are essentially the same as those we have laid out earlier for the pure-diffusion case.

8.2 An Example

For illustration, we consider the simplest setting where the productivity jump from H to L is permanent and irreversible, in that $\lambda_L = 0$. We set $\lambda_H = 0.1$ and choose the productivity levels to be $A^L = 0.18$ and $A^H = 0.2$. And all the other parameter values remain the same as those for the pure-diffusion case. Figure 8 plots the results. These results are broadly in line with what we have shown earlier. Importantly, note that the lower boundary for state H is to the left of that for state L, which makes intuitive sense, as the entrepreneur shall be able to borrow more in a more productive state, *ceteris paribus.*

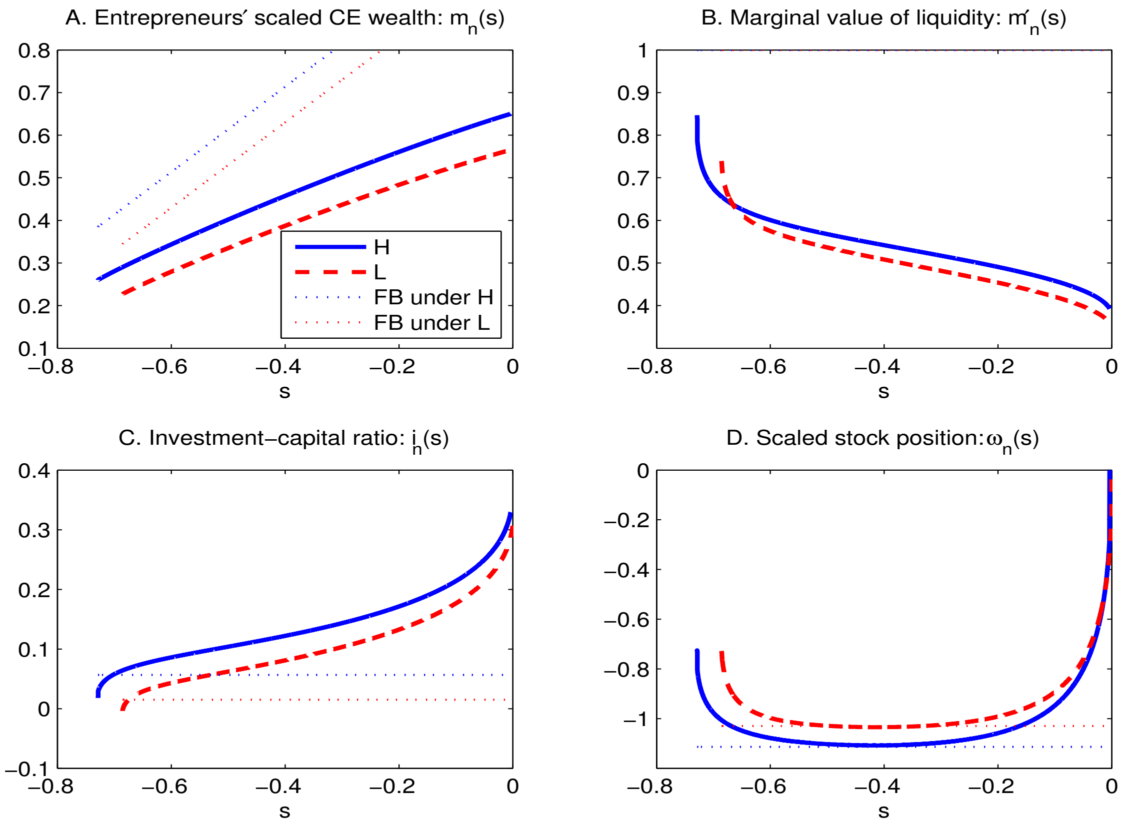


Figure 8: **The effect of firm's persistent productivity shocks.** Panels A, B, C, and D plot the entrepreneur's scaled certainty equivalent wealth $m_n(s)$, marginal (certainty equivalent) wealth of s, $m_n'(s)$, the investment-capital ratio $\mathfrak{i}_n(s)$, and optimal hedge via the risky asset $\omega_n(s)$, as functions of scaled liquidity s, respectively. Parameter values: $A^L = 0.18$, $A^H = 0.2$, $\lambda_L = 0$, and $\lambda_H = 0.1$.

Panel A of Figure 9 plots the entrepreneur's insurance demand $\pi_H(s)$ in state H against

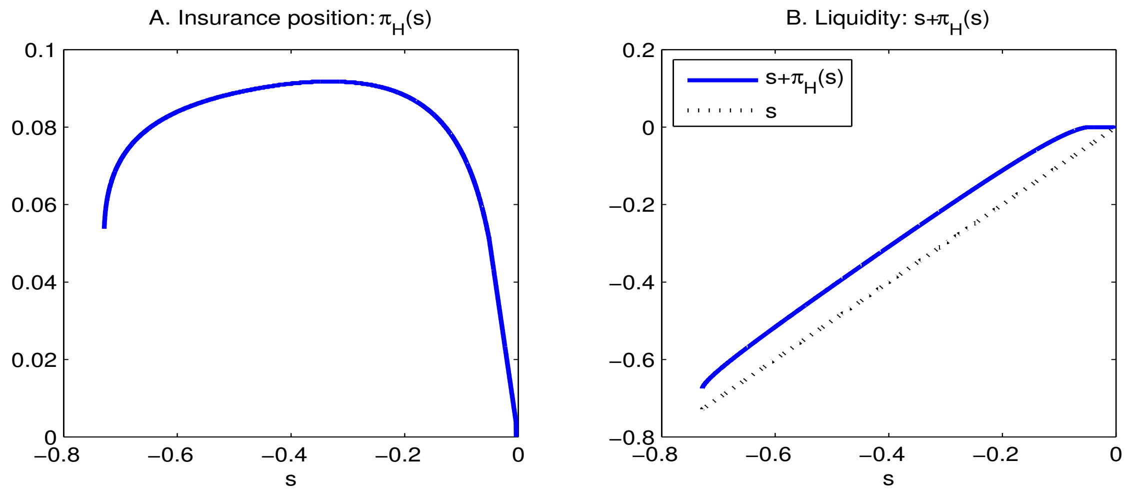


Figure 9: **The insurance position (against the change of productivity from state H to L) $\pi_H(s)$, and liquidity $s+\pi_H(s)$, as functions of scaled liquidity s.** Parameter values: $A^L = 0.18$, $A^H = 0.2$, $\lambda_L = 0$, and $\lambda_H = 0.1$.

the productivity change from state H to L. As we see for all levels of s, the entrepreneur pays a positive but time-varying insurance premium $\lambda_H \pi_H(s)$ per unit of time in state H to investors in order to receive a lump-sum insurance payment in the amount of $\pi_H > 0$ from investors at the moment when the productivity state switches from H to L. By doing so, the entrepreneur equates the marginal utility before and after the productivity changes whenever feasible. Interestingly, the insurance demand $\pi_H(s)$ is non-monotonic in s as it first increases in liquidity s and then decreases with s. The intuition is as follows. For a severely constrained entrepreneur whose s is close to the left boundary $\underline{s}$, the entrepreneur has limited funds to purchase insurance. Therefore, insurance $\pi_H(s)$ increases as s moves towards the origin turning less negative. As s becomes sufficiently close to the origin, the entrepreneur's demand for insurance decreases for the following reasons. First, the entrepreneurial firm has more liquidity to self insurance and hence demand for additional liquidity decreases. Second, the entrepreneur's decreasing marginal utility also suggests that the entrepreneur's demand for insurance is decreases with liquidity, *ceteris paribus.* Additionally, the investors' limited-liability constraint requires $\pi_H(s) \leq -s$, which in turns truncates the insurance demand. For these reasons, the insurance demand $\pi_H(s)$ is non-monotonic in liquidity s as shown in Panel A of Figure 9.

Panel B of Figure 9 plots the post-productivity-change liquidity level $s + \pi_H(s)$ immediately following the change of productivity from state H to L. Note that $s + \pi_H(s)$ is increasing in s, which makes intuitive sense. The higher the liquidity level in the current state H, the higher the post-productivity-change liquidity level $s + \pi_H(s)$ (ignoring the diffusion part.) We have already commented that $s + \pi_H(s) = 0$ for sufficiently high level of s because the demand for insurance reaches the constrained maximum level $\pi_H(s) = -s$, as $s \leq 0$ is required in both states H and L due to the investors' limited-liability condition. For firms that are not much in debt (with s sufficiently close to the origin,) the firm has sufficient liquidity and it is optimal to choose the constrained maximal insurance $\pi_H(s) = -s$ against the stochastic productivity change.

9 Conclusion

Our generalization of Hart and Moore (1994) to introduce risky human capital and cash flows, risk aversion of the entrepreneur, and ongoing consumption reveals the optimality of corporate liquidity and risk management for financially constrained firms. Most of the existing corporate security design literature has confined itself to showing that debt financing and credit line commitments are optimal financial contracts. By adding risky human capital and risk aversion for the entrepreneur, two natural assumptions, we show that corporate hedging policies are also an integral part of an optimal financial contract. When productivity shocks are persistent, we find that equilibrium default by the entrepreneur on his debt obligations is part of an optimal contract.

We have thus shown that the inalienability of human capital constraint naturally gives rise to a role for corporate liquidity and risk management, dimensions that are typically absent from existing macroeconomic theories of investment under financial constraints following Kiyotaki and Moore (1997).

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Appendices

To be added.